Every Random Choice Rule is Backwards-Induction Rationalizable *

Jiangtao Li† Rui Tang‡

October 25, 2016

Abstract

Motivated by the literature on random choice and in particular the random utility models, we extend the analysis in Bossert and Sprumont (2013) to include the possibility that players exhibit stochastic preferences over alternatives. We prove that every random choice rule is backwards-induction rationalizable.

*We are grateful to John Quah for his continuous and invaluable encouragement, support, and guidance. We thank Yi-Chun Chen, Xiao Luo, and Satoru Takahashi for helpful discussions. All remaining errors are our own.

†Department of Economics, National University of Singapore, jasonli1017@gmail.com

‡Department of Economics, National University of Singapore, rui.tang@u.nus.edu

1 Introduction

Bossert and Sprumont (2013) define a choice function as backwards-induction rationalizable "if there exists a finite perfect-information extensive-form game such that for each subset of alternatives, the backwards-induction outcome of the restriction of the game to that subset of alternatives coincides with the choice from that subset." Bossert and Sprumont (2013) then prove that every choice function is backwards-induction rationalizable. They focus on games where all players have *strict preferences* over the alternatives.

It is well known that individual choices exhibit variability, in both experimental and market settings; see for example, Sippel (1997), McFadden (2001), and Manzini, Mariotti, and Mittone (2010). The theoretical literature on random choice has focused largely on interpreting random choice as random utility maximization.[1] Motivated by the literature on random choice and in particular the random utility models (Block and Marschak (1960)), we extend the analysis in Bossert and Sprumont (2013) to include the possibility that players exhibit *stochastic preferences* over alternatives.

In a collective decision-making setting, if some player has a stochastic preference, then not surprisingly, the collective actions of the players might lead to a random outcome. We study the testable aspects of collective decision-making, allowing for *stochastic preferences* of the players. We extend the Bossert-Sprumont theorem, and prove that every random choice rule is backwards-induction rationalizable via stochastic preferences.

This note contributes to the emerging literature that applies the revealed preference approach to the study of collective decisions. Yanovskaya (1980), Sprumont (2000) and Galambos (2005) consider choice correspondences and Nash equilibria of normal-form games. Ray and Zhou (2001) and Ray and Snyder (2013) study Nash equilibria and sub-game perfect equilibria on extensive-form games. Xu and Zhou (2007) and Bossert and Sprumont (2013) examine when choice functions can be rationalized by an extensive-form game. Rehbeck (2014) and Xiong (2014) extend the Bossert-Sprumont theorem, and prove that every choice correspondence is backwards-induction rationalizable via *weak preferences*. In particular, the construction of the extensive-form game hinges upon a player who exhibits complete indifference among all alternatives.

[1] A random utility model is described by a probability measure over preference orderings, and the player selects the maximal alternative available according to the randomly assigned preference ordering; see for example, the seminal work of Block and Marschak (1960).

2 Definitions

Let X be a given finite universal set of alternatives, and denote by $\mathcal{P}(X)$ the collection of all nonempty subsets of X. The elements of $\mathcal{P}(X)$ are viewed as feasible sets that the players collectively choose an alternative from. We use $A, B, C, \ldots$ to denote alternative sets, and $x, y, z, \ldots$ to denote alternatives. Throughout the rest of the paper, unless it leads to confusion, we abuse the notation by suppressing the set delimiters, e.g., writing x rather than $\{x\}$. We use the following notational convention: $xy := x \cup y$.

A *choice function* is a map $f : \mathcal{P}(X) \to X$ such that $f(A) \in A$ for all $A \in \mathcal{P}(X)$. A *random choice rule* is a map $\rho : X \times \mathcal{P}(X) \to [0, 1]$ such that for all $A \in \mathcal{P}(X)$, we have i) $\sum_{x \in A} \rho(x, A) = 1$; and ii) $\rho(x, A) = 0$ for all $x \notin A$. The interpretation is that $\rho(x, A)$ denotes the probability that alternative x is chosen when the possible alternatives faced by the players are the alternatives in A.

In what follows, we present the relevant definitions and notations. Whenever possible, we keep the notations consistent with Bossert and Sprumont (2013) and Rehbeck (2014). We suggest that readers familiar with these two papers skip this section and return to it as needed.

Preference ordering. A *preference ordering* is a reflexive, complete, transitive and antisymmetric binary relation. We denote by $\mathcal{R}_A$ the set of all preference orderings on $A \in \mathcal{P}(X)$.

Precedence relation. Let $\prec$ be a transitive and asymmetric binary relation on a nonempty and finite set N. We say that $n \in N$ is a *direct predecessor* of $n' \in N$ if $n \prec n'$ and there is no $n'' \in N$ such that $n \prec n'' \prec n'$. Similarly, we say that $n \in N$ is a *direct successor* of $n' \in N$ if $n' \prec n$ and there is no $n'' \in N$ such that $n' \prec n'' \prec n$. The set of direct predecessors of $n \in N$ is denoted by $P(n)$. The set of direct successors of $n \in N$ is denoted by $S(n)$.

Tree. A *tree* Γ is given by a quadruple $(0, D, T, \prec)$, where the variables are defined as follows:

(i) the notation 0 is the *root*;

(ii) the variable D is a finite set of *decision nodes* such that $0 \in D$;

(iii) the variable T is a nonempty and finite set of *terminal nodes* such that $D \cap T = \emptyset$;

(iv) the notation $\prec$ is a transitive and asymmetric *precedence relation* on the set of all nodes $N = D \cup T$ such that:

(a) $P(0) = \emptyset$, and $|S(0)| \geq 1$;

(b) for all $n \in D \setminus \{0\}$, $|P(n)| = 1$, and $|S(n)| \geq 1$;

(c) for all $n \in T$, $|P(n)| = 1$, and $S(n) = \emptyset$.

Path. A *path in* Γ from a decision node $n \in D$ to a terminal node $n' \in T$ (of length $K \in \mathbb{N}$) is an ordered $(K+1)$ tuple $(n_0, n_1, \ldots, n_K) \in N^{|K+1|}$ such that $n_0 = n$, $\{n_{k-1}\} = P(n_k)$ for all $k \in \{1, 2, \ldots, K\}$, and $n_K = n'$.

Game. A *game* is a triple $G = (\Gamma, g, \pi)$ where

(i) $\Gamma = (0, D, T, \prec)$ is a tree;

(ii) $g : T \to X$ is an *outcome function* that maps each terminal node $n \in T$ to an alternative $g(n) \in X$;

(iii) π is a probability measure over the space of *preference assignment maps*, where each preference assignment map $R : D \to \mathcal{R}_X$ specifies for each decision node $n \in D$ a preference ordering $R(n) \in \mathcal{R}_X$. We denote by $\mathfrak{R}_{D,X}$ the space of all such preference assignment maps, and denote by $\Delta\mathfrak{R}_{D,X}$ the set of all probability measures over $\mathfrak{R}_{D,X}$. Formally, $\pi \in \Delta\mathfrak{R}_{D,X}$.

We focus on games in which the uncertainty on R resolves before any player makes a move, and the realization of R is commonly known among all the players. Let δ_R denote the degenerate measure at the preference assignment map R. For simplicity, sometimes we write $G = (\Gamma, g, R)$ rather than $G = (\Gamma, g, \delta_R)$.

Restriction of game. Fix a game $G = (\Gamma, g, \pi)$, we define the *restriction* of game G on $A \in \mathcal{P}(X)$ as $G|A = G_A = (\Gamma_A, g_A, \pi_A)$, where

(i) $0_A = 0$;

(ii) $D_A = \{n \in D : \text{ there exists } n' \in g^{-1}(A) \text{ and a path in } \Gamma \text{ from } n \text{ to } n'\}$;

(iii) $T_A = g^{-1}(A)$;

(iv) $\prec_A$ is the restriction of $\prec$ to $N_A = D_A \cup T_A$;

(iv) g_A is the restriction of g to T_A;

(v) $\pi_A \in \Delta\mathfrak{R}_{D_A,A}$ is the induced probability measure from $\pi \in \Delta\mathfrak{R}_{D,X}$. For any $R_A \in \mathfrak{R}_{D_A,A}$, $\pi_A(\{R_A\}) = \pi(\{R \in \mathfrak{R}_{D,X} : R_A \text{ is the restriction of } R \text{ to } D_A \text{ and } A\})$.

For any $R \in \mathfrak{R}_{D,X}$, we denote by R_A the restriction of R to D_A and A.

Solution concept. Denote by $\max(A; R^*)$ the unique best alternative in $A \in \mathcal{P}(X)$ according to the preference ordering $R^* \in \mathcal{R}_X$. Similarly, denote by $\min(A; R^*)$ the unique worst alternative in $A \in \mathcal{P}(X)$ according to the preference ordering $R^* \in \mathcal{R}_X$. Consider the game $G = (\Gamma, g, R)$. For each decision node $n \in D$, we denote by $e_n(\Gamma, g, R)$ the backwards-induction outcome of the subgame of $G = (\Gamma, g, R)$ rooted at n. We define the backwards-induction solution at node n as $e_n(\Gamma, g, R)$ in the usual way: first set $e_n(\Gamma, g, R) = g(n)$ for all $n \in T$, and then recursively set $e_n(\Gamma, g, R) = \max(\{e_{n'}(\Gamma, g, R) : n' \in S(n)\}; R(n))$ for all $n \in D$. We write $e(\Gamma, g, R) = e_0(\Gamma, g, R)$.

Backwards-induction rationalizable. A choice function f is *backwards-induction rationalizable* if there is a game $G = (\Gamma, g, R)$ such that

$$e(\Gamma_A, g_A, R_A) = f(A)$$

for any $A \in \mathcal{P}(X)$. We say that G is a *backwards-induction rationalization* of f or that G *backwards-induction rationalizes* f. A random choice rule ρ is *backwards-induction rationalizable* if there is a game $G = (\Gamma, g, \pi)$ such that

$$\pi(\{R \in \mathfrak{R}_{D,X} : e(\Gamma_A, g_A, R_A) = x\}) = \rho(x, A)$$

for any $x \in A \in \mathcal{P}(X)$. We say that G is a *backwards-induction rationalization* of ρ or that G *backwards-induction rationalizes* ρ.

3 Result

We prove that every random choice rule is backwards-induction rationalizable.

Theorem 1. *Every random choice rule is backwards-induction rationalizable.*

Proof. Without loss of generality, we assume that $|X| \geq 2$. For any given random choice rule ρ, we construct the tree $\Gamma = \{0, D, T, \prec\}$, the allocation function $g : T \to X$, and the probability measure $\pi \in \Delta\mathfrak{R}_{D,X}$, such that the game $G = (\Gamma, g, \pi)$ backwards-induction rationalizes ρ. Let $\bar{\mathcal{P}}(X) = \{A \in \mathcal{P}(X) : |A| \geq 2\}$.

Step 1. Building blocks.

Fix an arbitrary preference ordering $R^* \in \mathcal{R}_X$. By Bossert and Sprumont (2013, Theorem 1), every choice function is backwards-induction rationalizable. Consequently, for any $A \in \bar{\mathcal{P}}(X)$ and $x \in A$, there exists a game $G^{A,x} = (\Gamma^{A,x}, g^{A,x}, R^{A,x})$ where $\Gamma^{A,x} = (n^{A,x}, D^{A,x}, T^{A,x}, \prec^{A,x})$ such that

$$e(G^{A,x}|B) = \begin{cases} x; & \text{if } B = A; \\ \min(B; R^*); & \text{if } B \neq A. \end{cases} \tag{1}$$

Note that $n^{A,x}$ is the root of the tree $\Gamma^{A,x}$.

Step 2. Construction of the tree $\Gamma = (0, D, T, \prec)$.

(i) 0 is the root of the tree;

(ii) let $S(0) = \{n^A\}_{A \in \bar{\mathcal{P}}(X)}$. Clearly, $|S(0)| = |\bar{\mathcal{P}}(X)|$;

(iii) let $S(n^A) = \{n^{A,x}\}_{x \in A}$ for each $A \in \bar{\mathcal{P}}(X)$. Clearly, $|S(n^A)| = |A|$;

(iv) the set of decision nodes D is given by the union of $\{0\}$, $S(0)$, and the set of decision nodes $D^{A,x}$ of game $G^{A,x}$ for each $A \in \bar{\mathcal{P}}(X)$ and $x \in A$;

(v) the set of terminal nodes T is given by the union of the set of terminal nodes $T^{A,x}$ of game $G^{A,x}$ for each $A \in \bar{\mathcal{P}}(X)$ and $x \in A$;

(vi) the restriction of $\prec$ to $D^{A,x}$ coincides with $\prec^{A,x}$.

Figure 1 depicts the tree when there are three alternatives; that is, $X = \{x, y, z\}$.

Step 3. Construction of the outcome function g.

For any $n \in T$, we have that $n \in T^{A,x}$ for some $T^{A,x}$. The outcome function g is defined such that $g(n) = g^{A,x}(n)$.

Step 4. Construction of the probability measure $\pi \in \Delta\mathfrak{R}_{D,X}$.

We construct $\pi \in \Delta\mathfrak{R}_{D,X}$ such that:

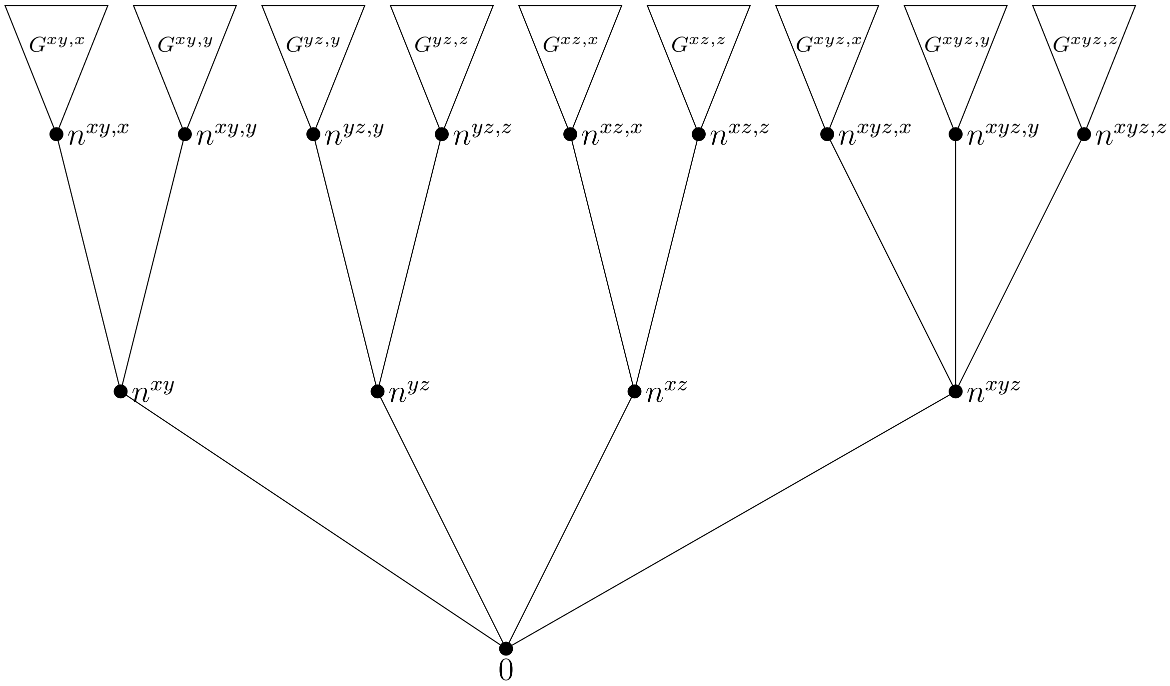

Figure 1: The tree when $X = \{x, y, z\}$.

(i) for the root 0, $\pi(\{R \in \mathfrak{R}_{D,X} : R(0) = R^*\}) = 1$;

(ii) for all $n \in D^{A,x}$, $\pi(\{R \in \mathfrak{R}_{D,X} : R(n) = R^{A,x}(n)\}) = 1$;

(iii) for all $x \in X$ and $A \in \bar{\mathcal{P}}(X)$, $\pi(\{R \in \mathfrak{R}_{D,X} : \max(A; R(n^A)) = x\}) = \rho(x, A)$.

The three conditions are restrictions imposed on the marginal distributions of π, and it is easy to see that such π exists.

Step 5. Backwards-induction rationalizable.

Lastly, we verify that $\pi(\{R \in \mathfrak{R}_{D,X} : e(\Gamma_A, g_A, R_A) = x\}) = \rho(x, A)$ for any $x \in A \in \mathcal{P}(X)$. We omit the trivial case when the feasible set is a singleton.

Fix $A \in \bar{\mathcal{P}}(X)$. For any preference ordering $R \in \mathfrak{R}_{D,X}$ in the support of $\pi \in \Delta\mathfrak{R}_{D,X}$, it follows from (1) that

$$e_{n^{A,x}}(\Gamma_A, g_A, R_A) = x;$$

for any $x \in A$, and

$$e_{n^{B,y}}(\Gamma_A, g_A, R_A) = \min(A; R^*)$$

for any $B \in \bar{\mathcal{P}}(X) \backslash \{A\}$ and $y \in B$.

Therefore,

$$e_{n^A}(\Gamma_A, g_A, R_A) = \max(A; R(n^A));$$

and

$$e_{n^B}(\Gamma_A, g_A, R_A) = \min(A; R^*)$$

for any $B \in \bar{\mathcal{P}}(X) \backslash \{A\}$.

By the construction in Step 4, $R(0) = R^*$. Hence,

$$e(\Gamma_A, g_A, R_A) = \max(A; R(n^A)). \tag{2}$$

Lastly,

$$\begin{aligned} \rho(x, A) &= \pi(\{R \in \mathfrak{R}_{D,X} : \max(A; R(n^A)) = x\}) \\ &= \pi(\{R \in \mathfrak{R}_{D,X} : e(\Gamma_A, g_A, R_A) = x\}), \end{aligned}$$

where the first equality follows from the construction in Step 4, and the second equality follows from (2). □

Remark 1. *We prove a stronger version of the Bossert-Sprumont theorem. Fix the alternative set X. Bossert and Sprumont (2013) show that, for any choice function f, there exists a tree $\Gamma = (0, D, T, \prec)$, an outcome function $g : T \to X$ and a preference assignment map $R : D \to \mathcal{R}_X$ such that $G = (\Gamma, g, R)$ backwards-induction rationalizes f. In contrast, we construct the tree $\Gamma' = (0', D', T', \prec')$ and outcome function $g' : T' \to X$ such that for any choice function f, there exists a preference assignment map $R' : D' \to \mathcal{R}_X$ such that $G' = (\Gamma', g', R')$ is a backwards-induction rationalization of f. In fact, the space of random choice rules defined on X is a mixture space,[2] and each random choice rule could be decomposed into a convex combination of choice functions. The randomness of collective choice behavior could be completely accounted by the stochastic preferences of the players, as proved in our theorem.*

[2]Given two random choice rules ρ and ρ', we define their convex combination $\alpha\rho + (1-\alpha)\rho'$ as $(\alpha\rho + (1-\alpha)\rho')(x, A) = \alpha\rho(x, A) + (1-\alpha)\rho'(x, A)$, for any $\alpha \in [0, 1]$.

References

Block, H. D., and J. Marschak (1960): "Random Orderings and Stochastic Theories of Responses," in *Contributions to Probability and Statistics*, pp. 97–132. Stanford, CA: Stanford University Press.

Bossert, W., and Y. Sprumont (2013): "Every Choice Function is Backwards-Induction Rationalizable," *Econometrica*, 81(6), 2521–2534.

Galambos, A. (2005): "Revealed Preference in Game Theory," mimeo. Northwestern University.

Manzini, P., M. Mariotti, and L. Mittone (2010): "Choosing Monetary Sequences: Theory and Experimental Evidence," *Theory and Decision*, 69(3), 327–354.

McFadden, D. (2001): "Economic Choices," *American Economic Review*, 91(3), 351–378.

Ray, I., and S. Snyder (2013): "Observable Implications of Nash and Subgame-Perfect Behavior in Extensive Games," *Journal of Mathematical Economics*, 49(6), 471–477.

Ray, I., and L. Zhou (2001): "Game Theory via Revealed Preferences," *Games and Economic Behavior*, 37(2), 415–424.

Rehbeck, J. (2014): "Every Choice Correspondence is Backwards-Induction Rationalizable," *Games and Economic Behavior*, 88, 207–210.

Sippel, R. (1997): "An Experiment on the Pure Theory of Consumer's Behaviour," *Economic Journal*, 107(444), 1431–1444.

Sprumont, Y. (2000): "On the Testable Implications of Collective Choice Theories," *Journal of Economic Theory*, 93(2), 205–232.

Xiong, S. (2014): "Every Choice Correspondence is Backwards-Induction Rationalizable," mimeo. University of Bristol.

Xu, Y., and L. Zhou (2007): "Rationalizability of Choice Functions by Game Trees," *Journal of Economic theory*, 134(1), 548–556.

Yanovskaya, E. (1980): "Revealed Preference in Noncooperative Games," *Mathematical Methods in the Social Sciences*, 24, 73–81, in Russian.