# Supplement to "Equivalence of Stochastic and Deterministic Mechanisms"

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This supplement contains several lemmas as the technical preparation for the proof of Proposition 1.[1]

## A Technical lemmas

If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are measurable spaces, then a measurable rectangle is a subset $A \times B$ of $X \times Y$, where $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ are measurable subsets of $X$ and $Y$, respectively. The "sides" $A$, $B$ of the measurable rectangle $A \times B$ can be arbitrary measurable sets; they are not required to be intervals. A discrete rectangle is a measurable rectangle such that each of its sides is a finite set.

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[1] These lemmas extend the corresponding mathematical results in Arkin and Levin (1972) from the special case with $I = 2$ and $\lambda$ the uniform distribution on $[0, 1] \times [0, 1]$ to the general setting in this paper. The corresponding mathematical results in Arkin and Levin (1972) were used to show the following result (see Theorem 2.3 therein): "Suppose that $f_1 \in L_1^\eta(X \times Y, \mathbb{R}^{l_1})$, $f_2 \in L_1^\eta(X \times Y, \mathbb{R}^{l_2})$ and $f_3 \in L_1^\eta(X \times Y, \mathbb{R}^{l_3})$, where $X = Y = [0, 1]$ and $\eta$ is the uniform distribution on $[0, 1] \times [0, 1]$. Let $A$ be the simplex $\{a = (a_1, \ldots, a_K) \colon \sum_{1 \le k \le K} a_k = 1, a_k \ge 0\}$. Given any measurable function $\alpha$ from $X \times Y$ to $A$, there exists another measurable function $\overline{\alpha}$ from $X \times Y$ to the vertices of the simplex $A$ such that $\int_{[0,1]} f_1(x, y)\alpha(x, y)\, \mathrm{d}y = \int_{[0,1]} f_1(x, y)\overline{\alpha}(x, y)\, \mathrm{d}y$, $\int_{[0,1]} f_2(x, y)\alpha(x, y)\, \mathrm{d}x = \int_{[0,1]} f_2(x, y)\overline{\alpha}(x, y)\, \mathrm{d}x$ and $\int_{[0,1]} \int_{[0,1]} f_3(x, y)\alpha(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_{[0,1]} \int_{[0,1]} f_3(x, y)\overline{\alpha}(x, y)\, \mathrm{d}x\, \mathrm{d}y$."

**Lemma A.1.** *Let $D$ be a Borel measurable subset of $V$, and $F \subseteq V$ a measurable rectangle with sides $Y_i \subseteq V_i$ of measure $l_i$, $i \in \mathcal{I}$. Assume that $\lambda(D \cap F) \geq (1-\epsilon)\lambda(F)$ for some $0 < \epsilon < 1$. Then for any $i$,*

$$\lambda_i\{v_i \in V_i \colon \lambda_{-i}(D_{v_i} \cap F_{v_i}) > (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})\} \geq (1-\sqrt{\epsilon})l_i.$$

*Proof.* Denote

$$\Gamma_i = \{v_i \in V_i \colon \lambda_{-i}(D_{v_i} \cap F_{v_i}) > (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})\}.$$

Let $\Gamma_i^C$ be the complement of $\Gamma_i$ in $V_i$. Then

$$\begin{aligned}
(1-\epsilon)\Pi_{1\leq j\leq I} l_j &= (1-\epsilon)\lambda(F) \\
&\leq \lambda(D \cap F) \\
&= (\int_{\Gamma_i} + \int_{\Gamma_i^C})\lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&= \int_{\Gamma_i} \lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) + \int_{\Gamma_i^C} \lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&\leq \int_{\Gamma_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon})\int_{\Gamma_i^C} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&= \sqrt{\epsilon}\int_{\Gamma_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon})\int_{Y_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&= \sqrt{\epsilon}\lambda_i(\Gamma_i) \cdot \Pi_{j\neq i} l_j + (1-\sqrt{\epsilon})\Pi_{1\leq j\leq I} l_j.
\end{aligned}$$

The first inequality holds due to the condition that $\lambda(D \cap F) \geq (1-\epsilon)\lambda(F)$. The second inequality is true since $\lambda_{-i}(D_{v_i} \cap F_{v_i}) \leq (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})$ for $v_i \in \Gamma_i^C$. All the equalities are just simple algebras. Rearranging the terms, we have

$$\lambda_i(\Gamma_i) \geq (1-\sqrt{\epsilon})l_i.$$

This completes the proof. □

**Lemma A.2.** *Let $D$ be a Borel measurable subset of $V$ with $\lambda(D) > 0$, $\tilde{i}_1, \ldots, \tilde{i}_I$ be positive natural numbers, and $0 < \epsilon < 1$ be sufficiently small such that $\epsilon' = \Pi_{1\leq j\leq I}\tilde{i}_j \cdot \epsilon < 1$ and $\Pi_{1\leq j\leq I}\tilde{i}_j \cdot \epsilon'^{\frac{1}{2^I}} < 1$.*

*Consider the system of measurable rectangles $F^{i_1,\ldots,i_I} = \prod_{1\leq j\leq I} Y_j^{i_j}$, where $1 \leq i_j \leq \tilde{i}_j$ and $Y_j^1, \ldots, Y_j^{\tilde{i}_j}$ are pairwise disjoint subsets on $V_j$ for $1 \leq j \leq I$ such that $\lambda(F^{i_1,\ldots,i_I} \cap D) \geq (1-\epsilon)\lambda(F^{i_1,\ldots,i_I})$. Then there exists a discrete rectangle $\{v_1^{i_1}, \ldots, v_I^{i_I}\}_{\{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I\}}$ such that*

1. *$(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D$ for $1 \leq i_j \leq \tilde{i}_j$ and $1 \leq j \leq I$;*

*2. for each* $1 \le j \le I$, $\{v_j^{i_j}\}$ *are different points for* $1 \le i_j \le \tilde{i}_j$.

*Proof.* First, we consider the set

$$\Gamma_1^{i_1,\dots,i_I} = \{v_1 \in Y_1^{i_1} : \lambda_{-1}(D_{v_1} \cap F_{v_1}^{i_1,\dots,i_I}) > (1-\sqrt{\epsilon'})\lambda_{-1}(F_{v_1}^{i_1,\dots,i_I})\}.$$

Denote $\Gamma_1^{i_1} = \cap_{1\le i_k \le \tilde{i}_k, 2\le k\le I}\Gamma_1^{i_1,\dots,i_I}$. We have

$$\begin{aligned}
\lambda_1(\Gamma_1^{i_1}) &= \lambda_1(Y_1^{i_1}) - \lambda_1\left(\cup_{1\le i_k\le \tilde{i}_k, 2\le k\le I}(Y_1^{i_1}\setminus \Gamma_1^{i_1,\dots,i_I})\right)\\
&\ge \lambda_1(Y_1^{i_1}) - \sum_{1\le i_k\le \tilde{i}_k, 2\le k\le I}\left(\lambda_1(Y_1^{i_1}) - \lambda_1(\Gamma_1^{i_1,\dots,i_I})\right)\\
&\ge \lambda_1(Y_1^{i_1}) - \sum_{1\le i_k\le \tilde{i}_k, 2\le k\le I}\left(\lambda_1(Y_1^{i_1}) - (1-\sqrt{\epsilon'})\lambda_1(Y_1^{i_1})\right)\\
&= \left(1 - \Pi_{2\le k\le I}\tilde{i}_k\cdot\sqrt{\epsilon'}\right)\lambda_1(Y_1^{i_1})\\
&> 0.
\end{aligned}$$

The second inequality holds due to Lemma A.1. We fix points $y_1^{i_1} \in \Gamma_1^{i_1}$ arbitrarily, as long as they are all distinct.

Second, let

$$\Gamma_2^{i_1,\dots,i_I} = \{v_2 \in Y_2^{i_2} : (\bigotimes_{3\le k\le I}\lambda_k)(D_{(y_1^{i_1},v_2)} \cap F_{(y_1^{i_1},v_2)}^{i_1,\dots,i_I}) > (1-\epsilon'^{\frac{1}{4}})(\bigotimes_{3\le k\le I}\lambda_k)(F_{(y_1^{i_1},v_2)}^{i_1,\dots,i_I})\}.$$

Since $y_1^{i_1} \in \Gamma_1^{i_1}$ for any $i_1$, we have $y_1^{i_1} \in \Gamma_1^{i_1,\dots,i_I}$ and

$$(\bigotimes_{2\le k\le I}\lambda_k)(D_{y_1^{i_1}} \cap F_{y_1^{i_1}}^{i_1,\dots,i_I}) > (1-\sqrt{\epsilon'})(\bigotimes_{2\le k\le I}\lambda_k)(F_{y_1^{i_1}}^{i_1,\dots,i_I}).$$

By Lemma A.1, we have

$$\lambda_2(\Gamma_2^{i_1,\dots,i_I}) \ge (1-\epsilon'^{\frac{1}{4}})\lambda_2(Y_2^{i_2}).$$

Denote $\Gamma_2^{i_2} = \cap_{1\le i_j\le\tilde{i}_j, j\ne 2}\Gamma_2^{i_1,\dots,i_I}$. We have

$$\begin{aligned}
\lambda_2(\Gamma_2^{i_2}) &= \lambda_2(Y_2^{i_2}) - \lambda_2\left(\cup_{1\le i_k\le\tilde{i}_k, k\ne 2}(Y_2^{i_2}\setminus\Gamma_2^{i_1,\dots,i_I})\right)\\
&\ge \lambda_2(Y_2^{i_2}) - \sum_{1\le i_k\le\tilde{i}_k, k\ne 2}\left(\lambda_2(Y_2^{i_2}) - \lambda_2(\Gamma_2^{i_1,\dots,i_I})\right)\\
&\ge \lambda_2(Y_2^{i_2}) - \sum_{1\le i_k\le\tilde{i}_k, k\ne 2}\left(\lambda_2(Y_2^{i_2}) - (1-\epsilon'^{\frac{1}{4}})\lambda_2(Y_2^{i_2})\right)\\
&= \left(1 - \Pi_{1\le k\le I, k\ne 2}\tilde{i}_k\cdot\epsilon'^{\frac{1}{4}}\right)\lambda_2(Y_2^{i_2})
\end{aligned}$$

$$> 0.$$

We fix points $y_2^{i_2} \in \Gamma_2^{i_2}$ arbitrarily, as long as they are all distinct, and are also different from $\{y_1^{i_1}\}$.

Repeating this procedure until $I-1$, we can find $y_k^{i_k} \in \Gamma_k^{i_k}$ for $1 \leq i_k \leq \tilde{i}_k$ and $1 \leq k \leq I-1$, where $\Gamma_k^{i_k} = \cap_{1\leq i_j \leq \tilde{i}_j, j\neq k} \Gamma_k^{i_1,\ldots,i_I}$ and $\lambda_k(\Gamma_k^{i_k}) > 0$. In particular,

$$\begin{aligned}\Gamma_{I-1}^{i_1,\ldots,i_I} =& \Big\{ v_{I-1} \in Y_{I-1}^{i_{I-1}} : \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})}) \\ & > (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})})\Big\}.\end{aligned}$$

Finally, consider the set

$$E^{i_I} = \cap_{1\leq i_k \leq \tilde{i}_k, 1\leq k \leq I-1} \left( D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I} \right).$$

Notice that $F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} = Y_I^{i_I}$ for any $i_1,\ldots,i_I$. Then

$$\begin{aligned}\lambda_I(E^{i_I}) &= \lambda_I \left( \cap_{1\leq i_k\leq \tilde{i}_k, 1\leq k\leq I-1} (D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I}) \right) \\ &= \lambda_I(Y_I^{i_I}) - \lambda_I \left( \cup_{1\leq i_k\leq \tilde{i}_k, 1\leq k\leq I-1} (Y_I^{i_I} \setminus D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) \right) \\ &\geq \lambda_I(Y_I^{i_I}) - \sum_{1\leq i_k\leq \tilde{i}_k, 1\leq k\leq I-1} \left( \lambda_I(Y_I^{i_I}) - \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I}) \right) \\ &= \lambda_I(Y_I^{i_I}) - \sum_{1\leq i_k\leq \tilde{i}_k, 1\leq k\leq I-1} \left( \lambda_I(Y_I^{i_I}) - \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) \right) \\ &> \lambda_I(Y_I^{i_I}) - \sum_{1\leq i_k\leq \tilde{i}_k, 1\leq k\leq I-1} \left( \lambda_I(Y_I^{i_I}) - (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) \right) \\ &= \lambda_I(Y_I^{i_I}) - \sum_{1\leq i_k\leq \tilde{i}_k, 1\leq k\leq I-1} \left( \lambda_I(Y_I^{i_I}) - (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(Y_I^{i_I}) \right) \\ &= \left( 1 - \Pi_{1\leq k\leq I-1} \tilde{i}_k \cdot \epsilon'^{\frac{1}{2^{I-1}}} \right) \lambda_I(Y_I^{i_I}) \\ &> 0.\end{aligned}$$

The second inequality holds since $y_{I-1}^{i_{I-1}} \in \Gamma_{I-1}^{i_{I-1}} \subseteq \Gamma_{I-1}^{i_1,\ldots,i_I}$, and hence

$$\lambda_I(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}) > (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})}).$$

Fix points $y_I^{i_I} \in E^{i_I}$ arbitrarily, as long as they are all different, and are different from $\{y_j^{i_j}\}_{1\leq j\leq I-1, 1\leq i_j\leq \tilde{i}_j}$. By the choice of $E^{i_I}$, $(y_1^{i_1},\ldots,y_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D$ for any $1 \leq i_j \leq \tilde{i}_j$ and $1 \leq j \leq I$. This completes the proof. □

Now we prove the last lemma.

**Lemma A.3.** *$\mathscr{E}$ is not dense in $L_1^\lambda(D,\mathbb{R})$.*[2] *In particular, there is a measurable function $d(v)$ with a finite set of values, which cannot be approximated in measure on $(D,\mathcal{B}(D),\lambda)$ by functions in $\mathscr{E}$.*

*Proof.* Let $g = \mathbf{1}_D$ be the indicator function of the set $D$, and $g_\delta(v) = \frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)} g \,\mathrm{d}\lambda$. By Lemma 4.1.2 in Ledrappier and Young (1985), $g_\delta \to g$ for $\lambda$-almost all $v \in \mathbb{R}^{lI}$ as $\delta \to 0$. Without loss of generality, we assume that this convergence result holds for each point of $D$ and the function $h$ is continuous on $D$.

Fix natural numbers $\tilde{i}_j$ satisfying the condition that $l \cdot \sum_{J\in\mathcal{J}}(\Pi_{j\in J}\tilde{i}_j) < \Pi_{1\le j\le I}\tilde{i}_j$. For any discrete rectangle $L = \{(v_1^{i_1},\ldots,v_I^{i_I}) \in D\colon 1 \le i_j \le \tilde{i}_j, 1 \le j \le I\}$, we associate a linear mapping $T_L$ from $\mathbb{R}^{\Pi_{1\le j\le I}\tilde{i}_j}$ to $\mathbb{R}^{l_0\cdot\sum_{J\in\mathcal{J}}(\Pi_{j\in J}\tilde{i}_j)}$:

$$T_L(w) = \{\sum_{j\notin J, 1\le i_j\le \tilde{i}_j} h(v_1^{i_1},\ldots,v_I^{i_I})\cdot w^{i_1,\ldots,i_I}\}_{1\le i_j\le\tilde{i}_j, j\in J, J\in\mathcal{J}},$$

where $l_0 = IKM+1$, $w$ is a vector with dimensions $\tilde{i}_1,\ldots,\tilde{i}_I$ and $w^{i_1,\ldots,i_I}$ is the corresponding component.

Fix a discrete rectangle $\overline{L} \subseteq D$ such that

- $\overline{L} = \{(\overline{v}_1^{i_1},\ldots,\overline{v}_I^{i_I}) \in D\colon 1 \le i_j \le \tilde{i}_j, 1 \le j \le I\}$;
- the rank of the mapping $T_{\overline{L}}$ is maximal, say $r$.

Consider the system of $\sum_{J\in\mathcal{J}}(\Pi_{j\in J}\tilde{i}_j)$ homogeneous linear equations with $\Pi_{1\le j\le I}\tilde{i}_j$ unknowns:

$$T_{\overline{L}}(w) = 0.$$

We take $r$ equations and $r$ unknowns for which the corresponding determinant is nonzero. Without loss of generality, we focus on this $r\times r$ matrix and denote it as $\overline{L}_s$, then $\det(\overline{L}_s) \neq 0$. For any discrete rectangle $L$, denote $L_s$ as the restriction of the vector generated by the operator $T_L$ onto the same matrix. Since $h$ is continuous, $\det(L_s) \neq 0$ for any discrete rectangle $L$ in a small open neighborhood of $\overline{L}$.

Let $w_{\overline{L}}$ be a nontrivial solution of the system corresponding to the discrete rectangle $\overline{L}$ in the sense that $T_{\overline{L}}(w_{\overline{L}}) = 0$. For any discrete rectangle $L \subseteq D$ such that $\det(L_s) \neq 0$, we provide a solution $w_L$ below such that $T_L(w_L) = 0$.

[2] Recall that $\mathscr{E}$ is defined in the proof of Proposition 1.

- Since $\det(L_s) \neq 0$, the rank of the system corresponding to the operator $T_L$ is at least $r$. Due to the choice of $\overline{L}$, the rank of the system corresponding to the operator $T_L$ is at most $r$, and hence is $r$. As a result, the equations that do not occur in the determinant $\det(L_s)$ are linear combinations of the $r$ equations that do.

- We focus on the $r$ equations that occur in the determinant $\det(L_s)$, and let $w_L^{i_1,\ldots,i_I} = w_{\overline{L}}^{i_1,\ldots,i_I}$ if the column corresponding to the unknown $w_L^{i_1,\ldots,i_I}$ does not occur in the determinant $\det(L_s)$.

- The remaining $r$ unknowns of $w_L^{i_1,\ldots,i_I}$, corresponding to the columns that occur in the determinant $\det(L_s)$, can be obtained by Cramer's rule.

By the above construction, it is obvious that $w_L$ depends continuously on the $r$ nodes of the discrete rectangle $L$ corresponding to the columns of $\det(L_s)$.

Pick numbers $d^{i_1,\ldots,i_I}$ subject to $\sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d^{i_1,\ldots,i_I} \cdot w_{\overline{L}}^{i_1,\ldots,i_I} = 1$. Consider the measurable rectangles

$$G^{i_1,\ldots,i_I} = \{v = (v_1,\ldots,v_I) \in \mathbb{R}^{lI} \colon |v_j - \overline{v}_j^{i_j}| \leq \delta,\ 1 \leq j \leq I\},$$

and

$$F^{i_1,\ldots,i_I} = \{v = (v_1,\ldots,v_I) \in V \colon |v_j - \overline{v}_j^{i_j}| \leq \delta,\ 1 \leq j \leq I\}.$$

Then for sufficiently small $\delta$, $\{G^{i_1,\ldots,i_I}\}$ are pairwise disjoint, and $\{F^{i_1,\ldots,i_I}\}$ are also pairwise disjoint.

By the first paragraph of this proof, $\frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)} \mathbf{1}_D \,\mathrm{d}\lambda \to \mathbf{1}_D(v)$ for each $v \in D$. Since $(\overline{v}_1^{i_1},\ldots,\overline{v}_I^{i_I}) \in D$, $\lambda(G^{i_1,\ldots,i_I} \cap D) \geq (1-\epsilon)\lambda(G^{i_1,\ldots,i_I})$ for sufficiently small $\delta$, where $\epsilon$ is given in the proof of Lemma A.2. Since $D$ is a subset of $V$, we have

$$\lambda(F^{i_1,\ldots,i_I} \cap D) = \lambda(G^{i_1,\ldots,i_I} \cap D) \geq (1-\epsilon)\lambda(G^{i_1,\ldots,i_I}) \geq (1-\epsilon)\lambda(F^{i_1,\ldots,i_I}).$$

In addition, since $\sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I}$ is continuous in the discrete rectangle, for sufficiently small $\delta$, $\sum_{1\leq i_j\leq \tilde{i}_j, 1\leq j\leq I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I} \geq \frac{1}{2}$ for

$$L = \{(v_1^{i_1},\ldots,v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D \colon 1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}.$$

To summarize, we pick $\delta > 0$ sufficiently small such that

1. $\lambda(F^{i_1,\ldots,i_I} \cap D) \geq (1-\epsilon)\lambda(F^{i_1,\ldots,i_I})$; and

2. $\sum_{1\le i_j\le \tilde{i}_j, 1\le j\le I} d^{i_1,\dots,i_I}\cdot w_L^{i_1,\dots,i_I}\ge \frac{1}{2}$ for any discrete rectangle

$$L=\{(v_1^{i_1},\dots,v_I^{i_I})\in F^{i_1,\dots,i_I}\cap D\colon 1\le i_j\le \tilde{i}_j, 1\le j\le I\}.$$

Let

$$d(v)=\begin{cases} d^{i_1,\dots,i_I}, & \text{if } v\in F^{i_1,\dots,i_I}\cap D;\\ 0, & \text{otherwise.}\end{cases}$$

If it could be approximated by functions in $\mathscr{E}$ on $(D,\mathcal{B}(D),\lambda)$ in measure, then there is a sequence $d_n(v)=h(v)\cdot\sum_{J\in\mathcal{J}}\psi_J^n(v_J)$ which converges to $d$ on some Borel measurable subset $C$ such that $\lambda(C)=\lambda(D)$.

By condition (1) above and Lemma A.2, there exists a discrete rectangle $L=\{(v_1^{i_1},\dots,v_I^{i_I})\}_{\{1\le i_j\le \tilde{i}_j, 1\le j\le I\}}$ such that $(v_1^{i_1},\dots,v_I^{i_I})\in F^{i_1,\dots,i_I}\cap C$ for $1\le i_j\le \tilde{i}_j$ and $1\le j\le I$. Since $\sum_{1\le i_j\le\tilde{i}_j, j\notin J} w_L^{i_1,\dots,i_I} h(v_1^{i_1},\dots,v_I^{i_I})=0$ for any $J\in\mathcal{J}$, we have

$$\begin{aligned}
&\sum_{1\le i_j\le\tilde{i}_j,1\le j\le I} d^{i_1,\dots,i_I}\cdot w_L^{i_1,\dots,i_I}\\
&=\lim_{n\to\infty}\sum_{1\le i_j\le\tilde{i}_j,1\le j\le I} d_n(v_1^{i_1},\dots,v_I^{i_I})\cdot w_L^{i_1,\dots,i_I}\\
&=\lim_{n\to\infty}\sum_{1\le i_j\le\tilde{i}_j,1\le j\le I}\left(h(v_1^{i_1},\dots,v_I^{i_I})\cdot\sum_{J\in\mathcal{J}}\psi_J^n(v_J^{i_J})\right)w_L^{i_1,\dots,i_I}\\
&=\lim_{n\to\infty}\sum_{1\le i_j\le\tilde{i}_j,1\le j\le I}\left\{\left(w_L^{i_1,\dots,i_I}h(v_1^{i_1},\dots,v_I^{i_I})\right)\cdot\sum_{J\in\mathcal{J}}\psi_J^n(v_J^{i_J})\right\}\\
&=\lim_{n\to\infty}\sum_{J\in\mathcal{J}}\sum_{1\le i_j\le\tilde{i}_j,1\le j\le I}\left\{\left(w_L^{i_1,\dots,i_I}h(v_1^{i_1},\dots,v_I^{i_I})\right)\cdot\psi_J^n(v_J^{i_J})\right\}\\
&=\lim_{n\to\infty}\sum_{J\in\mathcal{J}}\sum_{1\le i_j\le\tilde{i}_j,j\in J}\left\{\sum_{1\le i_j\le\tilde{i}_j,j\notin J} w_L^{i_1,\dots,i_I}h(v_1^{i_1},\dots,v_I^{i_I})\right\}\cdot\psi_J^n(v_J^{i_J})\\
&=0,
\end{aligned}$$

where $v_J^{i_J}$ denotes the vector $(v_j^{i_j})_{j\in J}$. However, $\sum_{1\le i_j\le\tilde{i}_j,1\le j\le I} d^{i_1,\dots,i_I}\cdot w_L^{i_1,\dots,i_I}\ge\frac{1}{2}$ by condition (2) above, which is a contradiction. As a result, the function $d$ cannot be approximated by functions in $\mathscr{E}$ on $(D,\mathcal{B}(D),\lambda)$ in measure. This completes the proof. $\square$

# References

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