Correlation-Robust Auction Design*

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December 11, 2021

Abstract

We study the design of auctions when the auctioneer has limited statistical information about the joint distribution of the bidders' valuations. More specifically, we consider an auctioneer who has an estimate of the marginal distribution of a generic bidder's valuation but does not have reliable information about the correlation structure. We analyze the performance of mechanisms in terms of the revenue guarantee, that is, the greatest lower bound of revenue across all joint distributions that are consistent with the marginals. A simple auction format, the second-price auction with no reserve price, is shown to be asymptotically optimal, as the number of bidders goes to infinity. For markets with a finite number of bidders, we (1) solve for the robustly optimal reserve price that generates the highest revenue guarantee among all second-price auctions with deterministic reserve prices, and (2) show that a second-price auction with a random reserve price generates the highest revenue guarantee among all standard dominant-strategy mechanisms.

KEYWORDS: Robust mechanism design, correlation, second-price auction, low reserve price, duality approach, optimal transport.

*We are indebted to Tilman Börgers, Ben Brooks, Gabriel Carroll, and Jeff Ely for stimulating discussions and detailed comments that substantially improved the paper. We thank Dirk Bergemann, Jimmy Chan, Yi-Chun Chen, Eddie Dekel, Songzi Du, Seungjin Han, Anton Kolotilin, Stephan Lauermann, Jonathan Libgober, Bart Lipman, George Mailath, Meg Meyer, Stephen Morris, Xiaosheng Mu, Zvika Neeman, Alessandro Pavan, Harry Di Pei, Eran Shmaya, Marciano Siniscalchi, Satoru Takahashi, Weijie Zhong, and participants at various seminars and conferences for helpful comments. Part of this paper was written while Li was visiting the Department of Economics at Northwestern University, and he would like to thank the institution for hospitality and support. This research is supported by Research Grants Council of Hong Kong (No. 14503318) and the Lee Kong Chian Fellowship. An earlier version of this paper was circulated under the title "Robustly Optimal Reserve Price." First version April 2019.

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1 Introduction

Traditional models in mechanism design make strong assumptions about the detailed knowledge of the designer in the economic environment. Subsequently, the theoretical conclusions can sometimes be fragile; mechanisms that are optimized to perform well when the assumptions are exactly true may still fail miserably in the much more frequent cases when the assumptions are untrue. The so-called Wilson doctrine holds that practical mechanisms should be designed without assuming that the designer has precise knowledge about the economic environment.

This paper studies a robust version of the single-unit auction problem where we relax the assumption about the auctioneer's knowledge in the payoff environment—the auctioneer has limited statistical information about the joint distribution of the bidders' valuations. In particular, we consider an auctioneer who has an estimate of the marginal distribution of a generic bidder's valuation but has non-Bayesian uncertainty about the correlation structure. Lacking the knowledge of the correlation structure, our auctioneer ranks mechanisms according to their revenue guarantee, that is, the greatest lower bound of revenue across all joint distributions that are consistent with the marginals.

Several motivations can be offered for considering the robustness to the correlation structure:

- First, while it is relatively easy to estimate the distribution of a generic bidder's valuation, it is significantly more difficult to estimate the joint distribution, which is a much higher-dimensional object; the computational and sampling complexity of learning the joint distribution is exponential in the number of bidders. In other words, obtaining an accurate statistical estimate of the joint distribution of bidders' values often requires the observation of unrealistically many examples of the joint value profiles.

- Second, besides the statistical aspect that the joint distribution is a much higher-dimensional object, there are many practical reasons why the joint distribution might be quite hard to observe and learn. For example, there are many instances in which the auctioneer cannot pin down the identities of the bidders (such as auctions that take place over the Internet or when bidders bid through proxies). In this case, the auctioneer has no way of estimating the correlation structure. It is reasonable to assume that each bidder has identical prior distribution, which can be deduced from an empirical study of a small random subgroup of the buyers' population.

- Third, given the known importance of the correlation structure (see for example Myerson (1981) and Crémer and McLean (1988)), understanding the robustness to the correlation structure is an especially useful exercise, at the very least in the sense of providing a robustness check for known mechanisms.[1]

- Fourth, while we could model the auctioneer's (lack of) knowledge of the payoff environment in many different ways, the correlation-robust framework seems to be a natural starting point. The source of the uncertainty, the correlation structure, is the same as that in Carroll (2017), who considers a multi-dimensional screening problem in which the seller knows the marginal distribution of the buyer's valuation for each good but does not know the joint distribution.[2]

- Finally, reserve prices observed in real-world auctions are substantially lower than the theoretically optimal ones (see for example Hasker and Sickles (2010)). Under the correlation-robust framework, for large markets, the robustly optimal mechanism is the simple and familiar second-price auction with no reserve price; for a finite number of bidders, we show that typically the auctioneer finds it optimal to use a low reserve price. Thus, both our analysis for large markets and a finite number of bidders could be perceived as supporting the use of a low reserve price from a novel robustness perspective.

Within this correlation-robust framework, our analysis focuses on the following three dimensions:

(1) The design of auctions in large markets.

(2) The robustly optimal reserve price that generates the highest revenue guarantee among all second-price auctions with deterministic reserve prices in markets with a finite number of bidders.

(3) The robustly optimal mechanism that generates the highest revenue guarantee among all standard dominant-strategy mechanisms in markets with a finite number of bidders.[3]

Methodologically, we adopt the duality approach (optimal transport in particular). This approach is explained in detail in our analysis of large markets, and is used throughout the paper.

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[1]For example, if the optimal mechanism derived under the independent private-value model also performs well under other correlation structures, then we would be comfortable using such a mechanism even without an accurate statistical estimate of the correlation structure. Example 1 shows that this is not the case.

[2]While the auctioneer is assumed to know the marginal distribution in the correlation-robust framework, our results for large markets does not depend on the knowledge of the marginal distribution; see Remark 2(b).

[3]We say that a mechanism is standard if bidders who do not have the highest bid do not get the object. We borrow the terminology of "standard" from Bergemann, Brooks, and Morris (2019) who define standard mechanisms in a similar manner.

To fix ideas and also to illustrate some of the motivations of our analysis, let us revisit the seminal paper of Myerson (1981) that studies optimal auction design in the independent private-value setting.

Example 1. In the independent private-value setting, Myerson (1981) shows that, under a regularity condition, the optimal mechanism can be implemented via a second-price auction with a reserve price (denoted r_M) that does not depend on the number of bidders. Suppose that each bidder's valuation is uniformly distributed on the $[0, 1]$ interval. Then $r_M = \frac{1}{2}$. For a thought experiment, suppose that there is a large number of bidders and the auctioneer needs to decide between two mechanisms: the second-price auction with reserve price $\frac{1}{2}$ and the second-price auction with no reserve price. We argue that the auctioneer should use the second-price auction with no reserve price. Consider two cases.

Case (1): If the bidders' valuations are indeed independent, then the second-price auction with reserve price $\frac{1}{2}$ is optimal and generates a strictly higher expected revenue than the second-price auction with no reserve price. However, as the ex post revenue differs only in the region in which at most one bidder has a valuation above $\frac{1}{2}$, the difference in the expected revenue of these two mechanisms is vanishingly small as the number of bidders gets larger.

Case (2): Now consider an alternative scenario in which the bidders' valuations are maximally positively correlated (the assumption of independent types is untrue). Regardless of the number of bidders (provided that there are at least two bidders), the second-price auction with no reserve price is the optimal mechanism and generates an expected revenue of $\frac{1}{2}$, whereas the second-price auction with reserve price $\frac{1}{2}$ only generates an expected revenue of $\frac{3}{8}$. □

While we only considered two particular correlation structures, the analysis already suggests that in large markets, the second-price auction with the optimally chosen reserve price under the independent private-value model is more vulnerable than the second-price auction with no reserve price. Intuitively, the optimality of the second-price auction with reserve price r_M in the independent private-value model depends on the intricate tradeoff of the following two events: (1) the largest valuation is less than r_M (so that the reserve price is not favorable); and (2) the largest valuation is weakly larger than r_M conditional on that the second largest valuation is less than r_M (so that the reserve price is favorable). Thus, the second-price auction with reserve price r_M may not perform well if the correlation structure is misspecified. In contrast, the second-price auction with no reserve price generates an expected revenue that is equal to the expectation of the second largest valuation regardless of the correlation structure. As such, one might expect that the second-price auction with no reserve price is a reasonable mechanism given non-Bayesian uncertainty about the correlation structure.

Indeed, our first result, Theorem 1, establishes the robust optimality of the second-price auction with no reserve price in large markets among all dominant-strategy mechanisms.[4,5] We show that the revenue guarantee of the second-price auction with no reserve price converges to the expectation of a generic bidder's valuation. Importantly, the expectation of a generic bidder's valuation is an *upper bound* of the highest revenue guarantee in our framework; our auctioneer could never rule out the maximally positive correlation as a candidate for the joint distribution, and the expectation of a generic bidder's valuation is the full surplus under this particular correlation structure.[6]

Although the robustness of a mechanism is a key concern, it is only one of several desiderata in practical mechanism design. Indeed, when selecting an auction format, the auctioneer might have to balance many different criteria. This perspective (of balancing multiple criteria) makes our result all the more appealing: besides having nice theoretical properties and being widely adopted in practice, the second-price auction with no reserve price is asymptotically optimal in the correlation-robust framework.[7] Our analysis supports the use of the second-price auction with no reserve price in large markets from a novel robustness perspective, complementary to existing reasons.

To show that the revenue guarantee of the second-price auction with no reserve price converges to the expectation of a generic bidder's valuation, we need to solve the minimization problem in which Nature minimizes the auctioneer's expected revenue by choosing a joint distribution that is consistent with the marginals. While this is a non-trivial task, due to the functional form of the ex post revenue function of the second-price auction with no reserve price, there is strong intuition about the properties of the worst-case correlation structure: if bidder i has the highest valuation and bidder j has the second highest valuation, then all the other bidders have the same valuation as

[4] Theorem 1 continues to hold even if the auctioneer uses a Bayesian mechanism, if we model the bidders' beliefs to be derived from the joint distribution; see Remark 1(a) for further discussion.

[5] One might think that the design of auctions in large markets is less interesting, on the ground that many auction formats including the second-price auction with any reserve price are optimal. While it is indeed the case that the choice of the reserve price does not matter under the assumption of independent values (since a large number of independent draws form some distribution would ensure that the second largest valuation approaches the upper bound of possible valuations), this is not the case in the correlation-robust framework, as illustrated in Example 1. The underlying logic of the optimality of the second-price auction in our correlation-robust framework and under the assumption of independent values is also drastically different.

[6] Abusing the terminology slightly, we refer to the expectation of a generic bidder's valuation as the full surplus in the correlation-robust framework hereafter.

[7] English auctions have a number of properties that make them attractive for practical purposes. They are weakly group strategy-proof, preserve the privacy of trading agents, endow the bidders with obviously dominant strategies, and limit the information that agents and the designer must acquire prior to the auction. English auctions are also widely adopted in practice. Besides these properties that are common to all English auctions, the English auction with no reserve price is efficient and does not demand the commitment power that the auctioneer commits to permanently withholding an unsold object off the market; see Liu, Mierendorff, Shi, and Zhong (2019). English auctions and second-price auctions are strategically equivalent in our setting. To economize on notation, we work with second-price auctions.

that of bidder j. This is because (1) the choice of Nature for the other bidders' valuation does not matter for the ex post revenue for this particular realization; (2) since Nature is bounded by the marginal consistency constraint, choosing the same valuation as that of bidder j provides the maximum flexibility for Nature to reduce the auctioneer's ex post revenue for other realizations.

This intuition leads us to consider a candidate correlation structure that has a natural economic interpretation as the maximally positive correlation conditionally on the existence of a strong bidder. Let F denote the distribution of a generic bidder. One bidder, whom the seller believes is equally likely to be any one of the bidders, is a strong bidder whose value is drawn from F conditional on that her valuation is weakly higher than some threshold. Every other bidder is a weak bidder whose value is drawn from F conditional on that their values are weakly less than this threshold. Furthermore, all the bidders' valuations are maximally positively correlated.

To show that the candidate correlation structure is indeed a worst-case correlation structure, we adopt a duality approach. This step of our analysis is closely related to the optimal transport theory (see for example Villani (2003)). To wit, Nature's minimization problem can be interpreted as an optimal transportation problem in which Nature seeks to implement the transportation at minimal cost. A transportation plan is a joint distribution that is consistent with the marginals, and Nature's cost function is the ex post revenue function of the auctioneer. While the literature of optimal transport focuses on the case of two random variables, we work with multiple random variables. To be rigorous and self-contained, we prove an n-dimensional generalization of the weak duality property in the Kantorovich duality theorem (see Villani (2003, Theorem 1.1.3)). This generalization is straightforward and follows from a modification of the original proof.

While Theorem 1 is a result on large markets, the second-price auction with no reserve price also performs well in small and moderate sized markets. We show (in Remark 1(d)) that for any marginal distribution, if there are n bidders, the difference of the full surplus and the revenue guarantee of the second-price auction with no reserve price is bounded above by $\frac{1}{n}$ (when the set of valuations is normalized to $[0, 1]$). For a numerical example, suppose that each bidder's valuation is uniformly distributed on the $[0, 1]$ interval. The revenue guarantee of the second-price auction with no reserve price is 75% of the full surplus with 4 bidders, and is 90% of the full surplus with 10 bidders.

The auctioneer could potentially do better using other mechanisms in markets with a finite number of bidders. For practical purposes, we first consider a familiar class of auction forms, second-price auctions with deterministic reserve prices, that are both theoretically appealing and widely adopted in practice.[8] Formally, we work with a maxmin optimization problem in which the auctioneer chooses a deterministic reserve price to maximize the worst-case expected revenue, where the worst case is taken over all joint distributions that are consistent with the marginals.[9] Theorem 2 solves for the robustly optimal reserve price that generates the highest revenue guarantee among all deterministic reserve prices for any finite number of bidders. We then focus on a class of dominant-strategy mechanisms that we call standard dominant-strategy mechanisms (that is, mechanisms such that bidders who do not have the highest bid do not get the object), including second-price auctions with random reserve prices. Theorem 3 shows that a second-price auction with a random reserve price generates the highest revenue guarantee among all standard dominant-strategy mechanisms.

The remainder of the introduction reviews the related literature. Section 2 presents our model. Section 3 and Section 4 show that the second-price auction with no reserve price is asymptotically optimal. Section 5 solves for the robustly optimal reserve price for any finite number of bidders. Section 6 shows that a second-price auction with a random reserve price generates the highest revenue guarantee among all standard dominant-strategy mechanisms. Section 7 concludes the paper.

1.1 Related literature

This paper joins the burgeoning literature of robust mechanism design.[10] A large body of papers focus on the case in which the designer does not have reliable information about the agents' hierarchies of beliefs about each other while assuming the knowledge of the payoff environment; see, for example, Bergemann and Morris (2005), Chung and Ely (2007), Chen and Li (2018), Yamashita and Zhu (2020), Bergemann, Brooks, and Morris (2016, 2017, 2019), Du (2018), Brooks and Du (2021), and Libgober and Mu (2020).[11]

The focus of this paper is on the uncertainty about the payoff environment, that is, the distribution of the bidders' valuations. More explicitly, our auctioneer has an estimate of the distribution of a generic bidder's valuation, but has non-Bayesian uncertainty about the correlation structure. Thus, the closest to our paper in terms of the source of

[8]We emphasize that it is important to understand the revenue guarantee of standard auction formats such as second-price auctions with reserve prices. While second-price auctions with reserve prices might not provide the highest revenue guarantee among all mechanisms, they are nevertheless one of the most common forms of auctioning an object and have many other desirable features aside from revenue guarantee.

[9]Ostrovsky and Schwarz (2016) present the results of a large-scale field experiment on reserve prices, and the data shows that reserve prices can play an important role in auction design.

[10]See Carroll (2019) for a recent survey on robust mechanism design and references therein.

[11]Börgers and Li (2019) propose a notion of strategic simplicity that can be interpreted as a form of robustness—the outcome implemented in strategically simple mechanism does not depend on higher-order beliefs.

uncertainty is Carroll (2017), who considers a multi-dimensional screening setting in which the seller knows the marginal distribution of the buyer's valuation for each good but does not know the joint distribution. Each mechanism is evaluated by its worst-case expected profit, over all joint distributions that are consistent with the known marginals. In this setting, Carroll (2017) shows that the optimal mechanism for the seller is simply to screen along each component separately. Following our work, Zhang (2021) also studies the design of auctions within the correlation-robust framework, and shows that a second-price auction with a random reserve price generates the highest revenue guarantee among all standard dominant-strategy mechanisms. His setting is different from ours, as the marginal distribution in his paper needs to satisfy a certain probability mass condition. Our results do not imply his result, and vice verse.

Several papers are similar in spirit to ours in that the auctioneer is assumed to have some limited information about payoff environment and evaluates mechanisms using the worst-case criterion. These papers assume that the auctioneer only knows some moment conditions (for example, the mean) that the marginal distribution needs to satisfy. Neeman (2003) considers an auctioneer who knows (a lower bound of) the mean of each bidder's valuation. He works with the notion of "effectiveness," which is the ratio of the revenue generated by the English auction and the benchmark of full-surplus extraction (this benchmark may not be feasible even for the optimal mechanism). In the screening environment, Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2018) study the revenue maximization problem of a seller who is partially informed about the distribution of the buyer's valuation, only knowing its first n moments. Their framework is general, as it covers an arbitrary number of moments. Koçyiğit, Iyengar, Kuhn, and Wiesemann (2020) study, among other settings, second-price auctions with reserve price when there are n ex ante symmetric bidders with a known lower bound for the mean. They characterize the optimal reserve price, and also show that randomized mechanisms yield strictly more revenue in this setting. Suzdaltsev (2020a) considers an auctioneer who knows that the bidders' valuations are independent draws from some unknown distribution F, and solves for the reserve price in a second-price auction to maximize worst-case expected revenue among all deterministic reserve prices under two specifications: (1) the seller knows the mean of F and an upper bound on values; (2) the seller knows the mean of F and an upper bound on its variance. He shows that it is optimal to set the reserve price to seller's own valuation. Suzdaltsev (2020b) considers an auctioneer who knows only the means and an upper bound for valuations. He shows that among all deterministic and dominant-strategy mechanisms, a linear version of Myersonian optimal auction generates the highest revenue guarantee. Che (2020) considers an auctioneer who only knows the mean of the marginal distribution of each bidder's valuation and the range, and shows that a second-price auction with an optimal, random reserve price obtains the optimal revenue guarantee within a broad class of mechanisms. As in our paper, the optimal reserve price in Che (2020) also converges to zero as the number of bidders goes to infinity. In comparison to these papers, the ambiguity set (the set of distributions that are perceived to be plausible) in our setting is smaller.

There is also a large literature in computer science that shows that simple mechanisms can perform reasonable well in a variety of settings. The most closely related to our work is Bei, Gravin, Lu, and Tang (2019) that consider the design of auctions in the correlation-robust framework. They use a different performance measure from the revenue guarantee. They focus on the sequential posted-price mechanism (SPM) and (among other results) show that SPM achieves a constant ($2\ln 4 + 2 \approx 4.78$) approximation to the optimal correlation-robust mechanism, that is, the revenue guarantee of the optimal sequential posted-price mechanism is weakly larger than $\frac{1}{2\ln 4+2}$ times the revenue guarantee of the optimal dominant-strategy mechanism.

Our paper is also related to the correlation neglect literature; see for example Levy and Razin (2015). Indeed, if our auctioneer ignores the possibility of a high correlation between the valuations of the bidders (which turns out to be true), the auctioneer might choose a mechanism that performs badly; see Example 1.

2 Preliminaries

2.1 Notation

For any real-valued vector $x \in \mathbb{R}^n$, we write $x(k)$ for the k-th largest element of the vector. For any set S, we denote by $|S|$ its cardinality. If Y is a measurable set, then ΔY is the set of all probability measures on Y. If Y is a metric space, then we treat it as a measurable space with its Borel σ-algebra.

2.2 The auction environment

An auctioneer seeks to sell a single indivisible object. There are $n \geq 2$ risk-neutral bidders competing for the object. We denote by $I = \{1, 2, \ldots, n\}$ the set of bidders and i a typical bidder. Each bidder i holds private information about her valuation of the object, which is modeled as a random variable v_i with cumulative distribution function F_i. We denote by V_i the set of possible valuations of bidder i. The set of possible valuation profiles is $V = \times_{i\in I} V_i$ with a typical element v. We write v_{-i} for a valuation profile of bidder i's opponents; that is, $v_{-i} \in V_{-i} = \times_{j\neq i} V_j$. Apart from their private information, all bidders are identical. Hereafter, we shall write F for the common cumulative distribution function.[12] Without loss of generality, we normalize the support of F to be $[0, 1]$. We

[12]Abusing notation slightly, we also use F to denote the probability measure that is consistent with the distribution F.

assume that F has a positive density f everywhere on the support.

The auctioneer has an estimate of the marginal distribution of a generic bidder's valuation, but has non-Bayesian uncertainty about the correlation structure. Any joint distribution is a plausible candidate as long as it is consistent with the marginals. We denote by

$$\Pi_n(F) = \Big\{\pi \in \Delta V : \forall i \in I, \forall A_i \subseteq V_i, \, \pi(A_i \times V_{-i}) = F(A_i)\Big\}$$

the collection of such joint distributions in the setting with n bidders and marginal distribution F. For ease of notation, we sometimes drop the dependence of $\Pi_n(F)$ on the number of bidders n and/or the marginal distribution F. As we focus on dominant-strategy mechanisms, we make no assumption about the bidders' beliefs. In particular, we do not assume that a common prior exists, nor that the bidders' and the auctioneer's beliefs are consistent.

2.3 Dominant-strategy mechanisms

We focus on dominant-strategy mechanisms. The revelation principle holds, and we restrict attention to direct mechanisms. A direct mechanism (q, t) consists of an allocation rule $q : V \to [0, 1]^n$ and a payment function $t : V \to \mathbb{R}^n$. Each bidder will report a valuation v_i, and based on the resulting profile of reports v, bidder i receives the object with probability $q_i(v)$ and pays $t_i(v)$ to the auctioneer.

A direct mechanism (q, t) is a dominant-strategy mechanism if for all $i \in I$, all $v \in V$, and all $v_i' \in V_i$,

$$v_i \, q_i(v) - t_i(v) \geq v_i \, q_i(v_i', v_{-i}) - t_i(v_i', v_{-i}),$$
$$v_i \, q_i(v) - t_i(v) \geq 0.$$

We denote by $\mathcal{M}_n$ the set of dominant-strategy mechanisms in the setting with n bidders, and we write M_n for a typical element of $\mathcal{M}_n$. For ease of notation, we sometimes drop the dependency of $\mathcal{M}_n$ and M_n on the number of bidders n.

We say that a dominant-strategy mechanism is standard if bidders who do not have the highest bid do not get the object.[13] Formally, let

$$\hat{\mathcal{M}}_n = \left\{ M_n \in \mathcal{M}_n \colon q_i(v_1, v_2, \ldots, v_n) = 0 \text{ if } v_i < \max_{1 \leq j \leq n} v_j \right\}$$

denote the set of standard dominant-strategy mechanisms in the setting with n bidders. The dependency on n is sometimes dropped.

[13] Thus, the outcome of any standard mechanism is envy-free.

We are interested in the auctioneer's expected revenue in the dominant-strategy equilibrium in which each bidder truthfully reports her valuation of the object. Let $REV(M,\pi) = \int_V \sum_{i\in I} t_i(v)\, d\pi(v)$. That is, we use $REV(M,\pi)$ to denote the auctioneer's expected revenue by using the mechanism M under the joint distribution π.

2.4 Second-price auctions with reserve prices

Second-price auctions with reserve prices play an important role in our analysis. In the second-price auction with reserve price r, each bidder i submits a bid $m_i \in \mathbb{R}_+$. Conditional on the submitted bids $m = (m_1, m_2, \dots, m_n)$, bidder i's probability of winning the object $q_i(m)$ and the payment from bidder i to the auctioneer $t_i(m)$ are given as follows:

$$q_i(m) = \begin{cases} \frac{1}{|W(m)|} & \text{if } i \in W(m) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t_i(m) = \begin{cases} \frac{\max(m(2),r)}{|W(m)|} & \text{if } i \in W(m) \\ 0 & \text{otherwise} \end{cases}$$

where $W(m) = \{i \in I :\, m_i = m(1),\, m_i \geq r\}$.

We are interested in the auctioneer's expected revenue in the dominant-strategy equilibrium in which each bidder submits a bid that is equal to her valuation of the object. For the second-price auction with reserve price r, let

$$REV(r,v) = \begin{cases} 0 & \text{if } v(1) < r; \\ r & \text{if } v(2) < r \leq v(1); \\ v(2) & \text{if } v(2) \geq r, \end{cases}$$

and let

$$REV(r,\pi) = \int_V REV(r,v)\, d\pi(v).$$

That is, we use $REV(r,v)$ to denote the auctioneer's ex post revenue by using the second-price auction with reserve price r when the realized valuation profile is v, and we use $REV(r,\pi)$ to denote the auctioneer's expected revenue by using the second-price auction with reserve price r under the joint distribution π.

2.5 Revenue guarantee as a criterion

We say that R is a revenue guarantee of mechanism M if for all $\pi \in \Pi$,

$$REV(M,\pi) \geq R.$$

We say that R is the revenue guarantee of mechanism M if it is a revenue guarantee and there is no higher revenue guarantee. Our auctioneer ranks mechanism according to the revenue guarantee. That is, the auctioneer solves the following maxmin optimization problem:

$$\sup_{M \in \mathcal{M}} \inf_{\pi \in \Pi} REV(M, \pi). \tag{Maxmin}$$

3 Large markets

We first consider the design of correlation-robust auctions in large markets. The auctioneer chooses a dominant-strategy mechanism to maximize the worst-case expected revenue, where the worst case is taken over all joint distributions that are consistent with the marginals.

We work with sequences of mechanisms $\{M_n\}_{n \geq 2}$ as the auctioneer may condition the choice of the mechanism on the number of bidders. That is, the auctioneer could use one mechanism M_2 when there are 2 bidders and use another mechanism M_{10} when there are 10 bidders. Abuse notation slightly, we use 0 (resp. r) to denote the second-price auction with no reserve price (resp. second-price auction with reserve price r) regardless of the number of bidders. When studying the asymptotic properties of a sequence of mechanisms, we write the second-price auction with no reserve price (resp. reserve price r) rather than the sequence of mechanisms where the designer uses the second-price auction with no reserve price (resp. reserve price r) for any number of bidders.

We say that a sequence of mechanisms $\{M_n\}_{n \geq 2}$ is asymptotically optimal if for any $\{M'_n\}_{n \geq 2}$, for any $\alpha < 1$, there exists N such that for all $n \geq N$, we have

$$\inf_{\pi \in \Pi} REV(M_n, \pi) > \alpha \inf_{\pi \in \Pi} REV(M'_n, \pi).$$

Theorem 1 below shows that a simple auction format, the second-price auction with no reserve price, is asymptotically optimal.

Theorem 1. *The second-price auction with no reserve price is asymptotically optimal.*

One may attempt to work with the maxmin optimization problem (Maxmin) directly. That is, we first ask, is there a systematic way of solving for the worst-case correlation structure for any dominant-strategy mechanism? In principle, if we have a way of identifying the worst-case correlation structure for any dominant-strategy mechanism, we could first calculate the worst-case expected revenue for any dominant-strategy mechanism, and then maximize the worst-case expected revenue (as a function of dominant-strategy mechanisms only) by choosing the mechanism. This approach is problematic, as it is far from clear (at least to us) what would be the worst-case correlation structure for any dominant-strategy mechanism.

We proceed indirectly. We first work with a simpler problem where we solve for the revenue guarantee of the second-price auction with no reserve price. We show that in the setting with n bidders, the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi\in\Pi} REV(0,\pi) = \frac{n}{n-1}\int_0^{F^{-1}(\frac{n-1}{n})} x\,dF(x).$$

Therefore, as $n\to\infty$,

$$\min_{\pi\in\Pi} REV(0,\pi) \to \int_0^1 x\,dF(x).$$

Importantly, $\int_0^1 x\,dF(x)$ can be interpreted as the *full surplus* in the correlation-robust framework. This is because our auctioneer could never rule out the maximally positive correlation (defined by randomly drawing $q\sim U[0,1]$ and taking $v_1 = v_2 = \ldots = v_n = F^{-1}(q)$) as a candidate for the joint distribution. Therefore, for whatever dominant-strategy mechanism that the auctioneer might use, be it a second-price auction with some reserve price or a more complex mechanism, the expectation of a generic bidder's valuation is an upper bound of the revenue guarantee of the mechanism.

The most involved part of our analysis is to solve for the revenue guarantee of the second-price auction with no reserve price. Formally, we need to solve the following minimization problem:

$$\inf_{\pi\in\Pi} REV(0,\pi).$$

This is a non-trivial task, as the space of joint distributions that are consistent with the marginals is large. For this step, we adopt the duality approach. We construct the dual maximization problem of the primal minimization problem and show that the optimal value of the maximization problem is weakly less than the optimal value of the minimization problem. That is, we establish a weak duality property. We then proceed to construct the primal variables and dual variables such that the value of the objective function of the minimization problem under the constructed primal variables and the value of the objective function of the maximization problem under the constructed dual variables are the same. This implies that the constructed primal variables are a solution to the primal minimization problem.[14]

Remark 1. (a) Theorem 1 shows that the second-price auction with no reserve price is

[14]For Theorem 1, it suffices to identify a lower bound of the worst-case revenue of the second-price auction with no reserve price that converges to the full surplus. While we are aware of alternative approaches of proving Theorem 1, we present the duality approach here as (1) the lower bound that we identify can be shown to be tight, which has the the added benefit of understanding the worst-case expected revenue and the worst-case correlation structure for the second-price auction with no reserve price for any finite number of bidders, and (2) this methodology will be used repeatedly throughout the paper, including the analysis of second-price auctions with (random) reserve prices for any finite number of bidders.

asymptotically optimal among all sequences of dominant-strategy mechanisms. Among other things, working with dominant-strategy mechanisms spares us the need to model the bidders' hierarchies of beliefs about each other. A Bayesian approach that makes detailed assumptions about the bidders' hierarchies of beliefs about each other goes against the spirit of our exercise. Having said that, if we model the bidders' beliefs to be derived from a common prior distribution that is the same as the true correlation structure, Theorem 1 continues to hold even if the auctioneer uses a Bayesian mechanism. This is because $\int_0^1 x\,dF(x)$ remains to be the full surplus in the Bayesian framework, as our auctioneer could never rule out the maximally positive correlation and that the agents' beliefs are derived from a common prior that is the maximally positive correlation.

(b) Notably, Theorem 1 does not rely on the knowledge of the marginal distribution (as long as the support is $[0,1]$). Even if the auctioneer does not possess the knowledge of the marginal distribution, the auctioneer finds it robustly optimal to use the second-price auction with no reserve price in large markets. More formally, let $\mathcal{F}$ be an arbitrary collection of marginal distributions, and let $\Pi_n(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} \Pi_n(F)$ denote the collection of joint distributions that the auctioneer considers plausible. Obviously, Theorem 1 still holds in this alternative setting.

(c) It is instructive to compare our model to that of Myerson (1981). As discussed in the introduction, the second-price auction with reserve price r_M (the optimal mechanism under the independence assumption) may not perform well if the correlation structure is misspecified. In contrast, we establish the robust optimality of the second-price auction with no reserve price in large markets. Intuitively, the second-price auction with no reserve price generates an expected revenue that is equal to the expectation of the second largest valuation regardless of the correlation structure. We formally show that in large markets, the worst case for the expectation of the second largest valuation is the expectation of a generic bidder's valuation. Another difference is that, while Myerson (1981) requires the regularity condition to establish the optimality of the second-price auction with some reserve price, our model does not impose any condition on the marginal distribution.

(d) Theorem 1 is a result on large markets. The second-price auction with no reserve price also performs well in small and moderate sized markets. For any F, if there are n bidders, the difference of the full surplus and the revenue guarantee of the second-price auction with no reserve price is bounded above by $\frac{1}{n}$. Indeed,

$$\begin{aligned}
&\int_0^1 x\,dF(x) - \inf_{\pi\in\Pi} REV(0,\pi) \\
&= \int_0^1 x\,dF(x) - \frac{n}{n-1}\int_0^{F^{-1}(\frac{n-1}{n})} x\,dF(x) \\
&= \int_{F^{-1}(\frac{n-1}{n})}^1 x\,dF(x) - \frac{1}{n-1}\int_0^{F^{-1}(\frac{n-1}{n})} x\,dF(x)
\end{aligned}$$

$$\leq \int_{F^{-1}(\frac{n-1}{n})}^{1} 1\, dF(x) = \frac{1}{n} \to 0.$$

For a numerical example, suppose that the marginal distribution F is the uniform distribution on the $[0, 1]$ interval. In the setting with n bidders, the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi \in \Pi} REV(0, \pi) = \frac{n}{n-1} \int_0^{F^{-1}(\frac{n-1}{n})} x\, dF(x) = \frac{n}{n-1} \int_0^{\frac{n-1}{n}} x\, dx = \frac{n-1}{2n},$$

whereas the full surplus is $\frac{1}{2}$. Thus, if there are n bidders, the revenue guarantee of the second-price auction with no reserve price is $\frac{n-1}{n}$ of the full surplus.

(e) There may be other sequences of mechanisms that are also asymptotically optimal.[15] In an earlier version of this paper He and Li (2020), we present a complementary result to Theorem 1 that among all sequences of standard mechanisms, the revenue guarantee of the second-price auction with no reserve price converges to the full surplus with the fastest rate of convergence.

4 Proof of Theorem 1

In this section, we show that in the setting with n bidders, the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi \in \Pi} REV(0, \pi) = \frac{n}{n-1} \int_0^{F^{-1}(\frac{n-1}{n})} x\, dF(x).$$

This, combined with the analysis in Section 3, establishes Theorem 1.

For the sake of clarity, we first consider the case in which there are only two bidders.

Observation 1. *Suppose that $n = 2$. A worst-case correlation structure for the second-price auction with no reserve price is the maximally negative correlation, defined by*

[15]The second-price auction with any positive reserve price r is not asymptotically optimal. Regardless of the number of bidders, the revenue guarantee of the second-price auction with a positive reserve price r cannot exceed its expected revenue under the maximally positive correlation, which is $\int_r^1 x\, dF(x)$ and is strictly bounded away from $\int_0^1 x\, dF(x)$.

randomly drawing $q \sim U[0,1]$ and taking[16,17]

$$v_1 = F^{-1}(q) \text{ and } v_2 = F^{-1}(1-q).$$

To see why this is a worst-case correlation structure, note that for the second-price auction with no reserve price, the auctioneer's ex post revenue function

$$REV(0, v) = v(2) = \min(v_1, v_2)$$

is a supermodular function. Since Nature chooses a joint distribution to minimize the expected value of a supermodular function, a worst-case correlation structure for the auctioneer is indeed the maximally negative correlation.[18] It follows that the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi \in \Pi} REV(0, \pi) = 2 \int_0^{F^{-1}(\frac{1}{2})} x \, dF(x).$$

Our analysis in the case of two bidders is particularly simple, as we could exploit the fact that the auctioneer's ex post revenue function $REV(0, v) = v(2) = \min(v_1, v_2)$ is a supermodular function. However, when there are more than two bidders, for the second-price auction with no reserve price, the ex post revenue function $REV(0, v) = v(2)$ is no longer a supermodular function. Thus, the worst-case correlation structure in the case of two bidders does not generalize in a straightforward manner. Nevertheless, for the second-price auction with no reserve price, since Nature's objective is to minimize the

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[16] There are other worst-case correlation structures. Consider the correlation structure defined by randomly drawing $s \sim U\{1,2\}$, $q \sim U[0, \frac{1}{2}]$, and taking $v_s = F^{-1}(q + \frac{1}{2})$ and $v_{-s} = F^{-1}(q)$. In words, a bidder, whom the auctioneer believes is equally likely to be any one of the bidders, is a strong bidder, the other bidder is a weak bidder, and the bidders' valuations are maximally positively correlated. Clearly, under this correlation structure, the auctioneer also obtains an expected revenue of $2 \int_0^{F^{-1}(\frac{1}{2})} x \, dF(x)$. Hence, this is also a worst-case correlation structure for the second-price auction with no reserve price.

[17] An equivalent but indirect way of defining the maximally negative correlation is as follows. The maximally negative correlation is the unique joint distribution such that

1. the probability concentrates on the following curve

$$L_0 : F(1) - F(v_2) = F(v_1) - F(0), \; v_1 \in [0, 1];$$

2. the joint distribution is consistent with the marginals.

While indirect, this alternative definition is somewhat more intuitive. Throughout the rest of the paper, we shall construct joint distributions indirectly.

[18] A function $g : V \to \mathbb{R}$ is supermodular if

$$g(v \vee v') + g(v \wedge v') \geq g(v) + g(v')$$

for all $v, v' \in V$, where $\vee$ denotes the component-wise maximum and $\wedge$ denotes the component-wise minimum. For detailed discussions on the ordering of joint distributions based on the integrals of supermodular functions, see for example Meyer and Strulovici (2012).

expectation of $v(2)$ by choosing a joint distribution that is consistent with the marginals, we have the following observation, which helps pin down a worst-case correlation structure when there are more than two bidders.

Observation 2. *Suppose that $n \geq 3$. Consider the thought experiment in which the values of $v(1)$ and $v(2)$ are fixed and Nature has the flexibility to choose $v(3), v(4), \ldots, v(n)$. For whatever values of $v(3), v(4), \ldots, v(n)$ that Nature chooses, the ex post revenue is $v(2)$, which is fixed by assumption.*

Observation 2 is about a specific scenario in which the values of $v(1)$ and $v(2)$ have been fixed. Thus, Nature's choice of $v(3), v(4), \ldots, v(n)$ does not matter. However, Nature's objective is not to minimize the ex post revenue for this particular realization of values, but to minimize the expectation of $v(2)$. Since Nature is constrained to choose a joint distribution that is consistent with the marginals, although the specific values of $v(3), \ldots, v(n)$ do not affect the ex post revenue for this particular realization, Nature would choose the other bidders' valuations to be as high as possible, that is, $v(3) = \ldots = v(n) = v(2)$. This choice gives Nature the maximum flexibility to minimize the expected revenue.

Motivated by these observations, we consider the following joint distribution, which is our candidate for a worst-case correlation structure. Define π^0 to be the unique joint distribution such that

1. the probability concentrates on n symmetric curves $L_0^1, L_0^2, \ldots, L_0^n$ where

$$L_0^i = \Big\{ v \in V : F(v_j) - F(0) = (n-1)(F(1) - F(v_i)),\ \forall j \neq i,\\ v_i \in [F^{-1}(\frac{n-1}{n}), 1] \Big\};$$

2. the joint distribution is consistent with the marginals.

The interpretation of the curve L_0^i is that in the region in which bidder i has the highest valuation, Nature puts probability in a way such that bidders other than i have the same valuation (motivated by Observation 2), and bidder i's valuation is maximally negatively correlated with the other bidders' valuation (motivated by Observation 1).

In what follows, we formally show that the correlation structure π^0 we construct above is a worst-case correlation structure for the second-price auction with no reserve price.

Proposition 1. *Suppose that $n \geq 2$.*[19] *Then*

$$\pi^0 \in \arg\min_{\pi \in \Pi} REV(0, \pi),$$

[19] This covers the case of two bidders as a special case.

and the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi \in \Pi} REV(0, \pi) = \frac{n}{n-1} \int_0^{F^{-1}(\frac{n-1}{n})} x \, dF(x).$$

Remark 2. As in the case of two bidders, there are other worst-case correlation structures. Consider the correlation structure defined by randomly drawing $s \sim U\{1, 2, \ldots, n\}$, $q \sim U[0, \frac{n-1}{n}]$, and taking $v_s = F^{-1}(\frac{q}{n-1} + \frac{n-1}{n})$ and $v_w = F^{-1}(q)$ for each $w \neq s$. In words, a bidder, whom the auctioneer believes is equally likely to be any one of the bidders, is a strong bidder, all the other bidders are weak bidders, and all the bidders' valuations are maximally positively correlated. It is easy to verify that the auctioneer obtains the same expected revenue under this correlation structure as that under π^0. Hence, this is also a worst-case correlation structure for the second-price auction with no reserve price.

To solve the minimization problem

$$\min_{\pi \in \Pi} REV(0, \pi), \tag{Primal-0}$$

we adopt a duality approach. We construct the dual maximization problem of the primal minimization problem and show that the optimal value of the maximization problem is weakly less than the optimal value of the minimization problem. We then proceed to construct the primal variables and dual variables such that the value of the objective function of the minimization problem under the constructed primal variables and the value of the objective function of the maximization problem under the constructed dual variables are the same. This implies that the constructed primal variables is a solution to the primal minimization problem.

Define $\mathbb{J}$ by

$$\mathbb{J} : L^1(F) \times L^1(F) \times \ldots \times L^1(F) \to \mathbb{R} \text{ and } \mathbb{J}(\mu_1, \mu_2, \ldots, \mu_n) = \sum_{i \in I} \int_{V_i} \mu_i(v_i) \, dF(v_i).$$

Consider the following dual maximization problem of the primal minimization problem:

$$\begin{aligned} \max_{\mu_1, \mu_2, \ldots, \mu_n \in L^1(F)} \quad & \mathbb{J}(\mu_1, \mu_2, \ldots, \mu_n) = \sum_{i \in I} \int_{V_i} \mu_i(v_i) \, dF(v_i) \\ \text{subject to} \quad & \text{for all } v \in V, \sum_{i \in I} \mu_i(v_i) \leq REV(0, v). \end{aligned} \tag{Dual-0}$$

The duality approach has a natural economic interpretation (see for example Villani (2003)). The optimal value of the primal minimization problem is the least cost for a planner that chooses a cost minimizing transport plan. The dual maximization problem may be interpreted as a decentralized solution. We can interpret $\mu_i(v_i)$ as the price paid per unit of mass to transport companies at the location v_i.[20] The optimal value of the dual maximization problem represents the maximal profit of transport companies with the profit being constrained by that the total cost paid to the transport sector for transporting one unit of goods $\sum_{i\in I} \mu_i(v_i)$ should not exceed the "if I do it myself" cost $REV(0, v)$.

Lemma 1. *The optimal value of the dual maximization problem (Dual-0) is weakly less than the optimal value of the primal minimization problem (Primal-0).*

Lemma 1 is essentially the n-dimensional generalization of the weak duality property in the Kantorovich duality theorem. The Kantorovich duality theorem establishes the strong duality in the case of two random variables. For our results, it suffices to prove the weak duality property. The extension to the case of n random variables is straightforward. To be self-contained, we present the short proof in Appendix A.

We are now ready to show that for the second-price auction with no reserve price, π^0 is a worst-case correlation structure. The proof proceeds as follows. Step (1) calculates the value of the objective function of the primal minimization problem under π^0. Step (2) constructs dual variables. Step (3) verifies that the value of the objective function of the dual maximization problem under the constructed dual variables is the same as the value of the objective function of the primal minimization problem under π^0.

Step (1). The value of the objective function of the primal minimization problem under π_0^* is

$$\frac{n}{n-1}\int_0^{c_n(0)} x\, dF(x)$$

where $c_n(0) = F^{-1}(\frac{n-1}{n})$ denotes the threshold for the reserve price 0.[21]

Step (2). For each $i \in I$, let

$$\mu_i(v_i) = \begin{cases} \frac{v_i}{n-1} - \frac{c_n(0)}{n(n-1)}, & \text{if } v_i < c_n(0); \\ \frac{c_n(0)}{n}, & \text{if } v_i \geq c_n(0). \end{cases}$$

It is easy to verify that these dual variables satisfy the constraints of the dual maximization problem. Indeed, since $\mu_i(v_i)$ is a weakly increasing function of v_i,

1. if $v(2) \geq c_n(0)$, then

$$\sum_{i\in I} \mu_i(v_i) \leq n\frac{c_n(0)}{n} = c_n(0) \leq v(2) = REV(0, v);$$

[20] More formally, $\mu_i(v_i)$ can be interpreted as the shadow cost of the primal minimization problem as one perturbs the marginal distribution F at v_i.

[21] In Section 5, we shall use the notation $c_n(r)$ to denote the threshold for the second-price auction with reserve price r.

2. if $v(2) < c_n(0)$, then

$$\sum_{i \in I} \mu_i(v_i) \leq (n-1)(\frac{v(2)}{n-1} - \frac{c_n(0)}{n(n-1)}) + \frac{c_n(0)}{n} = v(2) = REV(0, v).$$

Step (3). We now calculate the value of the objective function of the dual maximization problem under the constructed dual variables as follows:

$$\begin{aligned}
\mathbb{J}(\mu_1, \mu_2, \ldots, \mu_n) &= \sum_{i \in I} \int_{V_i} \mu_i(v_i)\, dF(v_i) \\
&= n \int_{V_1} \mu_1(v_1)\, dF(v_1) \\
&= n \int_0^{c_n(0)} \frac{v_1}{n-1} - \frac{c_n(0)}{n(n-1)}\, dF(v_1) + n \int_{c_n(0)}^1 \frac{c_n(0)}{n}\, dF(v_1) \\
&= \frac{n}{n-1} \int_0^{c_n(0)} v_1\, dF(v_1) - \frac{c_n(0)}{n-1} \int_0^{c_n(0)} 1\, dF(v_1) + c_n(0) \int_{c_n(0)}^1 1\, dF(v_1) \\
&= \frac{n}{n-1} \int_0^{c_n(0)} v_1\, dF(v_1),
\end{aligned}$$

where the last line follows from the definition of $c_n(0)$. This completes the proof of Proposition 1. Theorem 1 follows by taking $n \to \infty$.

5 Robustly optimal reserve price

Theorem 1 establishes the robust optimality of the second-price auction with no reserve price in large markets. In markets with a finite number of bidders, the auctioneer could potentially do better using other mechanisms. For practical purposes and also for tractability, in this section, we study an important class of auction forms, second-price auctions with reserve prices, that are both theoretically appealing and widely adopted in practice.

Our auctioneer chooses a reserve price to maximize the worst-case expected revenue, where the worst case is taken over all joint distributions that are consistent with the marginals. Formally, the auctioneer solves the following maxmin optimization problem:

$$\sup_{r \in [0,1]} \inf_{\pi \in \Pi} REV(r, \pi). \qquad \text{(Maxmin-r)}$$

We refer to the solution to this maxmin optimization problem as the robustly optimal reserve price.

The maxmin optimization problem (Maxmin-r) can be interpreted as a two-player zero-sum game. The two players are the auctioneer and Nature. The auctioneer first chooses a reserve price $r \in [0, 1]$. After observing the choice of the reserve price, Nature chooses a correlation structure $\pi \in \Pi$. The auctioneer's payoff is $REV(r, \pi)$, and Nature's payoff is $-REV(r, \pi)$.

It is not clear (at least to us) for an arbitrary reserve price what would be the worst-case correlation structure. To bypass this difficulty, we take an indirect approach. In the maxmin optimization problem (Maxmin-r), for each reserve price r, Nature can choose any joint distribution that is consistent with the marginals. This is not easy to work with, as the space of such joint distributions is very large. The novelty in our analysis is that we work with an auxiliary problem that has the interpretation that we impose a restriction on what Nature can do. For each reserve price r, we construct a particular correlation structure π^r that is consistent with the marginals. We can easily solve the following auxiliary problem:

$$\max_{r \in [0,1]} REV(r, \pi^r).$$ [22]

The auxiliary problem then corresponds to an extreme restriction on Nature's strategies in the sense that if the auctioneer chooses a reserve price r, Nature has no choice but to choose π^r. We show that the solution to this auxiliary problem is also the solution to the maxmin optimization problem (Maxmin-r).

The key step of our analysis is thus the construction of $\{\pi^r\}_{r \in [0,1]}$. The construction of $\{\pi^r\}_{r \in [0,1]}$ depends on the number of bidders and the marginal distribution, and will be made clear in the formal analysis. Before we move on to the formal analysis, we wish to provide a sketch of our analysis. The sketch highlights the requirements on $\{\pi^r\}_{r \in [0,1]}$ and should also make our approach more transparent.

In the first step, for each reserve price r, we explicitly construct a joint distribution π^r that is consistent with the marginals. At this stage, we do not know whether the joint distribution π^r that we construct is a worst-case correlation structure for the reserve price r. Nevertheless, since π^r is consistent with the marginals, the worst-case expected revenue of the reserve price r is weakly lower than its expected revenue under the correlation structure π^r. That is, for any r,

$$\inf_{\pi \in \Pi} REV(r, \pi) \leq REV(r, \pi^r).$$

In the second step, we solve the following auxiliary maximization problem:

$$\max_{r \in [0,1]} REV(r, \pi^r).$$

Let r^* denote a solution to the auxiliary maximization problem. Thus,

$$REV(r^*, \pi^{r^*}) \geq REV(r, \pi^r)$$

[22]Our construction of $\{\pi^r\}_{r \in [0,1]}$ ensures that a solution to this auxiliary problem exists.

for all r.

In the third step, we show that for the reserve price r^*, the correlation structure π^{r^*} is a worst-case correlation structure. Formally, we show that

$$REV(r^*, \pi^{r^*}) = \min_{\pi \in \Pi} REV(r^*, \pi).$$

Thus, for any r,

$$\inf_{\pi \in \Pi} REV(r, \pi) \leq REV(r, \pi^r) \leq REV(r^*, \pi^{r^*}) = \min_{\pi \in \Pi} REV(r^*, \pi).$$

It follows that r^* is a solution to the maxmin optimization problem (Maxmin-r).

For ease of exposition, we introduce one more notation. For any $r \in [0, 1]$ and any subset of bidders $S \subseteq I$, let

$$V^{r,S} = \{v \in V : \, v_i \geq r \text{ if and only if } i \in S\}.$$

In words, for any valuation profile $v \in V^{r,S}$, bidders in S have valuations weakly higher that r and bidders not in S have valuations lower than r. When S consists of a single bidder i, we write $V^{r,i}$ rather than $V^{r,\{i\}}$.

5.1 Two bidders

Suppose that there are only two bidders. Recall that for the second-price auction with no reserve price, a worst-case correlation structure is the maximally negative correlation (Observation 1). It is less clear what would be the worst-case correlation structure for an arbitrary reserve price r. Nevertheless, if Nature can only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$, we have a similar observation as in the case of the second-price auction with no reserve price.

Observation 3. *Fix an arbitrary reserve price $r \in [0, 1]$. In the constrained minimization problem in which Nature can only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$, a worst-case correlation structure is the unique joint distribution such that*

1. *in the region $V^{r,\{1,2\}}$, the probability concentrates on the following curve*

$$L_r : F(1) - F(v_2) = F(v_1) - F(r), \, v_1 \in [r, 1];$$

2. *in the region $V^{r,\emptyset}$, the probability concentrates on the following curve*

$$v_2 = v_1, \, v_1 \in [0, r);$$

3. the joint distribution is consistent with the marginals.

We denote this joint distribution by π^r (see Figure 1 for a graphical illustration of π^r in the case in which F is the uniform distribution on the $[0, 1]$ interval).

To see why this is a worst-case correlation structure when Nature is constrained to only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$, note that we can think of Nature's constrained minimization problem as two sub-problems, namely, the choice of the joint distribution in the region $V^{r,\emptyset}$ and the choice of the joint distribution in the region $V^{r,\{1,2\}}$. We can safely treat these two sub-problems separately, since these two choices do not interact with each other in terms of the consistency requirement. In the region $V^{r,\{1,2\}}$, the auctioneer's ex post revenue function is $REV(r, v) = v(2) = \min(v_1, v_2)$, which is a supermodular function. Therefore, our logic in Observation 1 applies here. In the region $V^{r,\emptyset}$, the joint distribution does not matter as long as it is consistent with the marginals, as the ex post revenue for any valuation profile in this region is zero. For concreteness, when constructing π^r, we pick the joint distribution such that the probability in the region $V^{r,\emptyset}$ concentrates on the curve $v_2 = v_1, v_1 \in [0, r)$. This particular choice plays no role in our analysis.

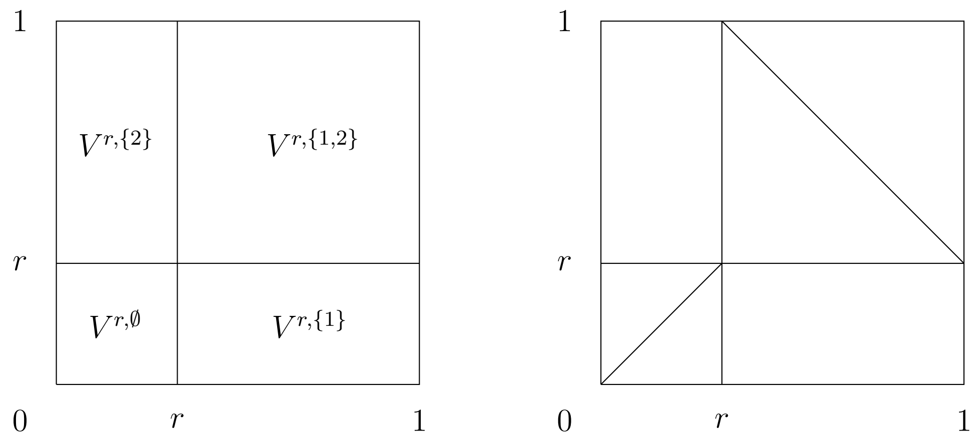

Figure 1: The figure on the left depicts the four regions given by $V^{r,\emptyset}$, $V^{r,1}$, $V^{r,2}$, and $V^{r,\{1,2\}}$. The figure on the right depicts the correlation structure π^r that we construct in the case in which F is the uniform distribution on the $[0, 1]$ interval.

While we have solved the constrained minimization problem in which Nature can only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$, our logic so far is incomplete for the purpose of identifying a worst-case correlation structure when Nature can choose any joint distribution that is consistent with the marginals, as Nature may want to allocate some probability to the regions $V^{r,1}$ and $V^{r,2}$.

Nevertheless, our analysis above leads us to consider an auxiliary maximization problem that we formulate below. Now that we have constructed the correlation structure

π^r for each $r \in [0,1]$, we can easily calculate the expected revenue of the reserve price r under π^r as follows:

$$REV(r, \pi^r) = \int_{[r,1]^2} v(2)\, d\pi^r(v) = 2\int_r^{c(r)} x\, dF(x),$$

where $c(r) = F^{-1}(\frac{1+F(r)}{2})$. Consider the following auxiliary maximization problem:

$$\max_{r\in[0,1]} REV(r, \pi^r) = 2\int_r^{c(r)} x\, dF(x). \tag{Max-2}$$

Proposition 2 below shows that the solution to the maximization problem (Max-2) is the robustly optimal reserve price.

Proposition 2. *Suppose that $n = 2$. Let r^* denote a solution to the maximization problem (Max-2). Then,*

$$\pi^{r^*} \in \arg\min_{\pi\in\Pi} REV(r^*, \pi).$$

This further implies that r^* is the robustly optimal reserve price and generates the highest revenue guarantee of $REV(r^*, \pi^{r^*})$.

Proposition 2 is a special case of Theorem 2, which solves for the robustly optimal reserve price in the general setting with n bidders. For this reason, we omit its proof.

The auxiliary maximization problem (Max-2) is easy to solve. In particular, by the first-order condition, we have $F(2r^*) = \frac{1+F(r^*)}{2}$. For any F, it is straightforward to solve for r^*. Example 2 below applies Proposition 2 to the case in which F is the uniform distribution on the $[0,1]$ interval.

Example 2 (Two bidders and uniform distribution)**.** Suppose that $n = 2$ and F is the uniform distribution on the $[0,1]$ interval. From our analysis above, r^* necessarily satisfies that $2r^* = \frac{1+r^*}{2}$. The robustly optimal reserve price is $\frac{1}{3}$ and generates the highest revenue guarantee of $\frac{1}{3}$. □

Remark 3. The key step in our analysis is the construction of the correlation structures $\{\pi^r\}_{r\in[0,1]}$. Proposition 2 shows that for r^*, the correlation structure π^{r^*} is a worst-case correlation structure. This suffices for our purpose of solving for the robustly optimal reserve price, sparing us the need to solve for the worst-case correlation structure for any $r \in [0,1]$.

5.2 n bidders

Next, we solve for the robustly optimal reserve price in the general setting with n bidders. Motivated by the worst-case correlation structure for the second-price auction with no reserve price (Proposition 1) and our analysis in Section 5.1, we construct the correlation structures $\{\pi^r\}_{r\in[0,1]}$ as follows. For each reserve price $r \in [0,1]$, we define π^r to be the unique joint distribution such that

1. it only puts positive probability in the regions $V^{r,\emptyset}$ and $V^{r,I}$;
2. in the region $V^{r,I}$, the probability concentrates on n symmetric curves $L_r^1, L_r^2, \ldots, L_r^n$ where

$$L_r^i = \Big\{v \in V^{r,I} : F(v_j) - F(r) = (n-1)(F(1) - F(v_i)),\ \forall j \neq i,$$
$$v_i \in [F^{-1}(\frac{(n-1)+F(r)}{n}), 1]\Big\};$$

3. in the region $V^{r,\emptyset}$, the probability concentrates on the following curve

$$v_i = v_1,\ \forall i \neq 1,\ v_1 \in [0,r];$$

4. the joint distribution is consistent with the marginals.

The interpretation of the curve L_r^i is that in the region in which every bidder's valuation is weakly higher than r and bidder i has the highest valuation, Nature puts probability in a way such that bidders other than i have the same valuation, and bidder i's valuation is maximally negatively correlated with the other bidders' valuation. In the region $V^{r,\emptyset}$, the joint distribution does not matter as long as it is consistent with the marginals, as the ex post revenue for any valuation profile in this region is zero. For concreteness, when constructing π^r, we pick the joint distribution such that the probability in the region $V^{r,\emptyset}$ concentrates on the curve $v_i = v_1$, $\forall i \neq 1$, $v_1 \in [0,r]$. This particular choice plays no role in our analysis.

Consider the following auxiliary maximization problem:

$$\max_{r\in[0,1]} REV(r,\pi^r) = \int_{[r,1]^n} v(2)\, d\pi^r(v) = \frac{n}{n-1}\int_r^{c_n(r)} x\, dF(x) \tag{Max-n}$$

where $c_n(r) = F^{-1}(\frac{(n-1)+F(r)}{n})$. Theorem 2 below shows that the solution to the maximization problem (Max-n) is the robustly optimal reserve price.

Theorem 2. *Suppose that there are n bidders. Let r_n^* denote a solution to the maximization problem (Max-n). Then,*

$$\pi^{r_n^*} \in \arg\min_{\pi\in\Pi} REV(r_n^*, \pi).$$

This further implies that r_n^* is the robustly optimal reserve price and generates the highest revenue guarantee of $REV(r_n^*, \pi^{r_n^*})$.

It suffices to show that for the reserve price r_n^*, $\pi^{r_n^*}$ is a worst-case correlation structure. Since r_n^* is a solution to the maximization problem (Max-n), for any reserve price r,

$$\inf_{\pi\in\Pi} REV(r,\pi) \leq REV(r,\pi^r) \leq REV(r^*,\pi^{r^*}) = \min_{\pi\in\Pi} REV(r^*,\pi).$$

Thus, r_n^* is the robustly optimal reserve price and generates the highest revenue guarantee of $REV(r_n^*, \pi^{r_n^*})$. To show that $\pi^{r_n^*}$ is a worst-case correlation structure for r_n^*, as in the case of the second-price auction with no reserve price, we adopt the duality approach. The proof can be found in Appendix B.

The auxiliary maximization problem (Max-n) is easy to solve. In particular, by the first-order condition, we have $F(nr_n^*) = F(\frac{(n-1)+F(r_n^*)}{n})$. Example 3 below illustrates how to apply Theorem 2 to the case in which F is the uniform distribution on the $[0,1]$ interval.

Example 3 (n bidders and uniform distribution)**.** Suppose that there are n bidders and F is the uniform distribution on the $[0,1]$ interval. From our analysis above, r_n^* necessarily satisfies that $nr_n^* = \frac{(n-1)+r_n^*}{n}$. The robustly optimal reserve price is $r_n^* = \frac{1}{n+1}$ and generates the highest revenue guarantee of $\frac{n}{2(n+1)}$. □

Theorem 2 solves for the robustly optimal reserve price for any finite number of bidders. We now briefly discuss the case in which the number of bidders is large. We first revisit the example where the marginal distribution is the uniform distribution on $[0,1]$.

Example 4 (large n and uniform distribution)**.** *Suppose that there are n bidders and F is the uniform distribution on the $[0,1]$ interval. As $n\to\infty$, the robustly optimal reserve price $r_n^* = \frac{1}{n+1} \to 0$, and the revenue guarantee $\frac{n}{2(n+1)} \to \frac{1}{2}$ which is the expectation of a generic bidder's valuation.*

These features generalize to any marginal distribution. Indeed, Theorem 2 has immediate implications as follows.

Corollary 1. *For any marginal distribution F,*

1. *$r_n^* < \frac{1}{n}$ for any n;*
2. *$\lim_{n\to\infty} r_n^* \to 0$;*
3. *$\lim_{n\to\infty} REV(r_n^*, \pi^{r_n^*}) \to \int_0^1 x\, dF(x)$; and*
4. *The second-price auction with no reserve price is asymptotically optimal.*

The first statement follows from the first-order condition of the auxiliary maximization problem (Max-n) that

$$F(nr_n^*) = F(\frac{(n-1)+F(r_n^*)}{n}) < 1.$$

The second statement then trivially follows. By Theorem 2,

$$REV(r_n^*, \pi^{r_n^*}) = \frac{n}{n-1}\int_{r_n^*}^{c_n(r_n^*)} x\,dF(x) \to \int_0^1 x\,dF(x)$$

as $n \to \infty$. The fourth statement follows from the observation that $\int_0^1 x\,dF(x)$ is the full surplus in our framework.

6 Standard dominant-strategy mechanisms

Section 5 focuses on second-price auctions with deterministic reserve prices. In this section, we extend our analysis to the case in which the auctioneer is allowed to use any standard dominant-strategy mechanism (that is, mechanisms such that bidders who do not have the highest bid do not get the object), including second-price auctions with random reserve prices. The auctioneer solves the following maxmin optimization problem:

$$\sup_{M\in\hat{\mathcal{M}}}\inf_{\pi\in\Pi} REV(M,\pi).$$

We show that a random reserve price generates the highest revenue guarantee among all standard dominant-strategy mechanisms.

Let $\mathcal{G}$ denote the set of all cumulative distribution functions on the $[0,1]$ interval. For any random reserve price G, let

$$REV(G,v) = \int_0^1 REV(r,v)\,dG(r),$$

and let

$$REV(G,\pi) = \int_V REV(G,v)\,d\pi(v).$$

That is, we use $REV(G,v)$ to denote the auctioneer's ex post revenue by using the random reserve price G when the realized valuation profile is v, and we use $REV(G,\pi)$ to denote the auctioneer's expected revenue by using the random reserve price G under the joint distribution π.

We make the following assumption: $xf(x)$ is weakly increasing in x. In words, this assumption says that the density function does not decrease too fast. We first present the following technical lemma, which is a consequence of the assumption that $xf(x)$ is weakly increasing in x.

Lemma 2. *Fix a distribution F such that $xf(x)$ is weakly increasing in x. Let $\psi(x) := x - \frac{1-F(x)}{f(x)}$, and let*

$$\gamma(x) := 1 - F(x) - \frac{1}{n-1}x^{-\frac{n}{n-1}}\int_0^x y^{\frac{n}{n-1}}f(y)\,dy.$$

Then,

1. $\lim_{x\to 0} xf(x) = 0$;
2. *there exists a unique* $b^* \in (0,1)$ *such that* $\psi(b^*) = 0$; *and*
3. *there exists* $x \in [b^*, 1]$ *such that* $\gamma(x) = 0$.

Let $\bar{b}_F$ be such that $\bar{b}_F \in [b^*, 1]$ and $\gamma(\bar{b}_F) = 0$. Let

$$G_F^*(r) = \bar{b}_F^{-\frac{1}{n-1}} r^{\frac{1}{n-1}}$$

with support $[0, \bar{b}_F]$.

Theorem 3. *Suppose that there are* n *bidders and each bidder's valuation is distributed according to* F *where* $xf(x)$ *is weakly increasing in* x. *Then,* G_F^* *is optimal among all standard dominant-strategy mechanisms, and generates the highest revenue guarantee of* $n\bar{b}_F(1 - F(\bar{b}_F))$.

Remark 4. *As the number of bidders gets large,* $\bar{b}_F$ *converges to 1,* $G_F^*(r) = \bar{b}_F^{-\frac{1}{n-1}} r^{\frac{1}{n-1}}$ *converges to the Dirac measure on zero, and the highest revenue guarantee is*

$$n\bar{b}_F(1 - F(\bar{b}_F)) = n\bar{b}_F \frac{1}{n-1} \bar{b}_F^{-\frac{n}{n-1}} \int_0^{\bar{b}_F} y^{\frac{n}{n-1}} f(y)\, dy,$$

which converges to $\int_0^1 x\, dF(x)$.

The logic of the proof is as follows. We first show that $n\bar{b}_F(1 - F(\bar{b}_F))$ is a lower bound of the revenue guarantee of the random reserve price G_F^*, that is,

$$\inf_{\pi\in\Pi} REV(G_F^*, \pi) \geq n\bar{b}_F(1 - F(\bar{b}_F)).$$

This step uses the duality approach (Lemma 1), where we explicitly construct the dual variables. We then construct a specific correlation structure π_F^* under which the optimal standard dominant-strategy mechanism generates the expected revenue of $n\bar{b}_F(1 - F(\bar{b}_F))$, that is,

$$\max_{M\in\hat{\mathcal{M}}} REV(M, \pi_F^*) = n\bar{b}_F(1 - F(\bar{b}_F)).$$

This implies that $n\bar{b}_F(1 - F(\bar{b}_F))$ is an upper bound of the revenue guarantee of any standard dominant-strategy mechanism. Thus,

$$\inf_{\pi\in\Pi} REV(G_F^*, \pi) \geq n\bar{b}_F(1 - F(\bar{b}_F)) = \max_{M\in\hat{\mathcal{M}}} REV(M, \pi_F^*) \geq \sup_{M\in\hat{\mathcal{M}}} \inf_{\pi\in\Pi} REV(M, \pi).$$

The proof of Theorem 3 is contained in Appendix D. For the sake of clarity, in what follows, we illustrate the proof using the example of two agents and uniform distribution.

Since we fix the marginal distribution to be the uniform distribution, for ease of notation, we drop the dependence on F. Since $n = 2$ and $F(x) = x$, we have $\bar{b} = \frac{3}{4}$ and

$$G^*(r) = \frac{4}{3}r$$

with support $[0, \frac{3}{4}]$.

We calculate the ex post revenue of the auctioneer as follows:

$$REV(G^*, v) = \begin{cases} \int_0^{v(2)} v(2)\ \mathrm{d}G^*(r) + \int_{v(2)}^{v(1)} r\ \mathrm{d}G^*(r) = \frac{2}{3}v(1)^2 + \frac{2}{3}v(2)^2, & \text{if } v(1) \leq \frac{3}{4}; \\ \int_0^{v(2)} v(2)\ \mathrm{d}G^*(r) + \int_{v(2)}^{\frac{3}{4}} r\ \mathrm{d}G^*(r) = \frac{2}{3}v(2)^2 + \frac{3}{8}, & \text{if } v(2) \leq \frac{3}{4} < v(1); \\ v(2), & \text{if } v(2) > \frac{3}{4}. \end{cases}$$

Let

$$u(x) = \begin{cases} \frac{2}{3}x^2, & \text{if } 0 \leq x \leq \frac{3}{4}; \\ \frac{3}{8}, & \text{if } \frac{3}{4} < x \leq 1. \end{cases}$$

One can easily verify that

$$REV(G^*, v) \geq \sum_{i \in I} u(v_i)$$

for all $v \in V$. It follows that for any $\pi \in \Pi$,

$$REV(G^*, \pi) \geq \int_V \sum_{i \in I} u(v_i)\ \mathrm{d}\pi(v) = \sum_{i \in I} \int_0^1 u(x)\ \mathrm{d}x = \frac{3}{8}.$$

It remains to show that there exists a correlation structure under which the optimal standard dominant-strategy mechanism generates an expected revenue of $\frac{3}{8}$. Consider the correlation structure π^* induced by the following symmetric density function (see Figure 2):

$$\eta^*(v) = \begin{cases} \frac{2v(2)}{3v(1)^2} & \text{if } 0 < v(2) \leq v(1) < \frac{3}{4}, \\ \frac{32v(2)}{9} & \text{if } 0 < v(2) < \frac{3}{4} < v(1) < 1, \\ 0 & \text{if } \frac{3}{4} < v(2) \leq v(1) < 1, \\ 1 & \text{if } v_i \in \{0, \frac{3}{4}, 1\} \text{ for some } i. \end{cases}$$

It is straightforward to verify that $\pi^* \in \Pi$.

For any density function η, let $\eta_i(v_i|v_{-i})$ denote the density of v_i conditional on v_{-i}, and let

$$\phi_i(v) = \begin{cases} v_i - \frac{1 - \int_0^{v_i} \eta_i(s|v_{-i})\ \mathrm{d}s}{\eta_i(v_i|v_{-i})} & \text{if } \eta(v) > 0, \\ 0 & \text{if } \eta(v) = 0 \end{cases}$$

denote the conditional virtual value of bidder i at the value profile v. It is straightforward

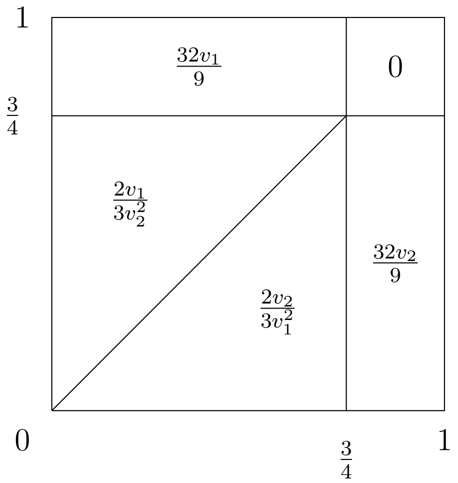

Figure 2: The figure depicts the density function η^* that we construct in the case of two bidders and uniform distribution.

to calculate that, under η^*, the conditional virtual value of agent i is

$$v_i - \frac{1 - (1 - \frac{2v_j}{3v_i})}{\frac{2v_j}{3v_i^2}} = 0$$

in the region in which $0 < v_j < v_i < \frac{3}{4}$ and is

$$v_i - \frac{1 - (1 - \frac{8v_j}{9} + \frac{32v_j}{9}(v_i - \frac{3}{4}))}{\frac{32v_j}{9}} = 2v_i - 1 > 0$$

in the region in which $0 < v_j < \frac{3}{4} < v_i < 1$. Using standard arguments (of calculating conditional virtual values and performing pointwise maximization subject to the constraint that bidders who do not have the highest value do not get the object) à la Myerson (1981), one can show that under the correlation structure π^*, the optimal standard dominant-strategy mechanism generates an expected revenue of

$$2\int_{\frac{3}{4}}^{1}\int_{0}^{\frac{3}{4}} \frac{32v_2}{9} \cdot (2v_1 - 1)\, \mathrm{d}v_2\, \mathrm{d}v_1 = \frac{3}{8}.$$

7 Conclusion

We consider a robust version of the single-unit auction problem in which the auctioneer has an estimate of the marginal distribution of a generic bidder's valuation but has non-Bayesian uncertainty about the correlation structure. A simple auction format, the second-price auction with no reserve price, is shown to be asymptotically optimal. In settings with a finite number of bidders, we study second-price auctions with deterministic reserve prices and solve for the robustly optimal reserve price that generates the highest revenue guarantee among all deterministic reserve prices. We show that typically the auctioneer finds it optimal to use a low reserve price. Both our analysis for large markets and a finite number of bidders could be perceived as supporting the use of a low reserve price from a novel robustness perspective.

We also show that among all standard dominant-strategy mechanisms, a second-price auction with a random reserve price generates the highest revenue guarantee. Extending the analysis beyond standard dominant-strategy mechanisms is an important question for further research. Further research might also consider additional restrictions on the joint distributions that the auctioneer perceives plausible. While classical papers such as Myerson (1981) consider one extreme formulation of the single-unit auction problem in the sense that the auctioneer knows the exact correlation structure, we consider the other extreme formulation in the sense that the auctioneer has no additional information besides the marginal distribution. It might be fruitful to investigate settings in which the auctioneer has some additional information besides the marginals, such as the knowledge that the bidders' valuations are positively correlated.

A Proof of Lemma 1

It suffices to show that for any feasible dual variables $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ of the dual maximization problem and any feasible primal variables π of the primal minimization problem, the value of the objective function of the maximization problem under μ is weakly less than the value of the objective function of the minimization problem under π. As we shall see below, this follows immediately from the feasibility constraint.

Let π be feasible variables of the primal minimization problem. That is, for all $i \in I$ and for all measurable sets $A_i \in V_i$,

$$\pi(A_i \times V_{-i}) = F(A_i). \tag{1}$$

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ be feasible variables of the dual maximization problem. That is, for all $v \in V$,

$$\sum_{i \in I} \mu_i(v_i) \leq REV(0, v). \tag{2}$$

Thus, we have

$$\begin{aligned}
\mathbb{J}(\mu) &= \sum_{i \in I} \int_{V_i} \mu_i(v_i)\, dF(v_i) \\
&= \sum_{i \in I} \int_{V} \mu_i(v_i)\, d\pi(v)
\end{aligned}$$

$$
\begin{aligned}
&= \int_V \sum_{i \in I} \mu_i(v_i)\, d\pi(v) \\
&\le \int_V REV(0, v)\, d\pi(v) \\
&= REV(0, \pi),
\end{aligned}
$$

where the second line follows from (1) and the fourth line follows from (2).

B Proof of Theorem 2

It suffices to show that for the reserve price r_n^*, $\pi^{r_n^*}$ is a worst-case correlation structure. The proof proceeds as follows. We first show that r_n^* necessarily satisfies

$$
F(nr_n^*) = \frac{(n-1) + F(r_n^*)}{n}.
$$

We then show that for any r such that $F(nr) = \frac{(n-1)+F(r)}{n}$, π^r is a worst-case correlation structure.

The first step is straightforward. This requirement on r_n^* is an immediate implication of the first-order condition. Consider the maximization problem:

$$
\max_{r \in [0,1]} REV(r, \pi^r) = \frac{n}{n-1} \int_r^{c_n(r)} x\, dF(x).
$$

By the first-order condition, we have $\frac{dREV(r,\pi^r)}{dr} = \frac{n}{n-1} f(r)(\frac{c_n(r)}{n} - r)$. Let

$$
\begin{aligned}
R_n :=& \{r \in [0,1] : \frac{n}{n-1} f(r)(\frac{c_n(r)}{n} - r) = 0\} \\
=& \{r \in [0,1] : F(nr) = \frac{(n-1) + F(r)}{n}\}
\end{aligned}
$$

denote the set of stationary points. Since the first-order derivative has a positive value at $r = 0$ and has a negative value at $r = 1$, the maximization problem has an interior solution. Thus, it must be that $r_n^* \in R_n$.

In what follows, we show that for any reserve price $r \in R_n$, π^r is a worst-case correlation structure. That is, π^r is a solution to the following minimization problem:

$$
\min_{\pi \in \Pi} REV(r, \pi). \tag{Primal-r}
$$

We adopt a duality approach. We construct the dual maximization problem (Dual-r) of the primal minimization problem (Primal-r) as follows:

$$\max_{\mu_1,\mu_2,\ldots,\mu_n \in L^1(F)} \quad \mathbb{J}(\mu_1,\mu_2,\ldots,\mu_n) = \sum_{i\in I}\int_{V_i} \mu_i(v_i)\,dF(v_i) \tag{Dual-r}$$
$$\text{subject to} \quad \text{for all } v \in V,\ \sum_{i\in I}\mu_i(v_i) \le REV(r,v).$$

As in the proof of Theorem 1, one can easily show that the optimal value of the maximization problem (Dual-r) is weakly less than the optimal value of the minimization problem (Primal-r).

We are now ready to show that for any reserve price $r \in R_n$, π^r is a worst-case correlation structure. Step (1) calculates the value of the objective function of the primal minimization problem (Primal-r) under π^r. Step (2) constructs dual variables and calculates the value of the objective function of the dual maximization problem (Dual-r) under the constructed dual variables. Step (3) verifies that these two values are the same for any $r \in R_n$.

Step (1). The value of the objective function of the primal minimization problem (Primal-r) under π^r is

$$\frac{n}{n-1}\int_r^{c_n(r)} x\,dF(x)$$

where $c_n(r) = F^{-1}(\frac{(n-1)+F(r)}{n})$.

Step (2). For each $i \in I$, let

$$\mu_i(v_i) = \begin{cases} 0, & \text{if } v_i < r; \\ \frac{1}{n-1}(v_i - r), & \text{if } r \le v_i < nr; \\ r, & \text{if } v_i \ge nr. \end{cases}$$

It is easy to verify that these dual variables satisfy the constraints of the dual maximization problem (Dual-r). Indeed, since $\mu_i(v_i)$ is a weakly increasing function of v_i,

1. if $v(2) \ge nr$, then $\sum_{i\in I}\mu_i(v_i) \le nr \le v(2) = REV(r,v)$;
2. if $v(1) \ge nr > v(2) \ge r$, then
$$\sum_{i\in I}\mu_i(v_i) \le r + (n-1)\frac{1}{n-1}(v(2)-r) = v(2) = REV(r,v);$$
3. if $v(1) \ge nr$ and $r > v(2)$, then $\sum_{i\in I}\mu_i(v_i) = r = REV(r,v)$;
4. if $nr > v(1) \ge v(2) \ge r$, then
$$\sum_{i\in I}\mu_i(v_i) \le \frac{1}{n-1}(nr-r) + (n-1)\frac{1}{n-1}(v(2)-r) = v(2) = REV(r,v);$$

5. if $nr > v(1) \geq r > v(2)$, $\sum_{i \in I} \mu_i(v_i) = \frac{1}{n-1}(v(1) - r) < r = REV(r, v)$;
6. if $r > v(1)$, then $\sum_{i \in I} \mu_i(v_i) = 0 = REV(r, v)$.

We now calculate the value of the objective function of the dual maximization problem (Dual-r) under the constructed dual variables as follows:

$$\begin{aligned}
\mathbb{J}(\mu_1, \mu_2, \ldots, \mu_n) &= \sum_{i \in I} \int_{V_i} \mu_i(v_i)\, dF(v_i) \\
&= n \int_{V_1} \mu_1(v_1)\, dF(v_1) \\
&= n \int_r^{nr} \frac{1}{n-1}(v_1 - r)\, dF(v_1) + n \int_{nr}^1 r\, dF(v_1) \\
&= \frac{n}{n-1} \int_r^{nr} v_1\, dF(v_1) - \frac{n}{n-1} \int_r^{nr} r\, dF(v_1) + n \int_{nr}^1 r\, dF(v_1). \qquad (3)
\end{aligned}$$

Step (3). Recall that $R_n = \{r \in [0,1] : F(nr) = \frac{(n-1)+F(r)}{n}\}$. Thus, for any $r \in R_n$,

$$c_n(r) = F^{-1}(\frac{(n-1) + F(r)}{n}) = nr.$$

The value of the objective function of the primal minimization problem (Primal-r) under π^r is

$$\frac{n}{n-1} \int_r^{c_n(r)} x\, dF(x) = \frac{n}{n-1} \int_r^{nr} x\, dF(x).$$

The value of the objective function of the dual maximization problem (Dual-r) under the constructed dual variables is also

$$\frac{n}{n-1} \int_r^{nr} x\, dF(x),$$

since the last two terms in (3) cancel off. This completes the proof that for any $r \in R_n$, π^r is a solution to the primal minimization problem (Primal-r).

C Proof of Lemma 2

1. Suppose that $\lim_{x \to 0} xf(x) = c > 0$. Since $xf(x)$ is weakly increasing in x, for any $x > 0$, we have that $xf(x) \geq c$ and $f(x) \geq \frac{c}{x}$. But then $F(x) \geq \int_0^x \frac{c}{y}\, dy = \infty$ for any $x > 0$. We have a contradiction.

2. Let $\rho(x) := xf(x) - (1 - F(x))$. Since $xf(x)$ is weakly increasing in x, $\rho(x)$ is increasing in x. Since $\lim_{x \to 0} \rho(x) < 0$ and $\eta(1) > 0$, there exists a unique b^* such that

$$\rho(x) \begin{cases} < 0, x < b^*; \\ = 0, x = b^*; \\ > 0, x > b^*. \end{cases}$$

Since $\psi(x) = \frac{\rho(x)}{f(x)}$, we have that

$$\psi(x) \begin{cases} < 0, x < b^*; \\ = 0, x = b^*; \\ > 0, x > b^*. \end{cases}$$

3. We show that (1) $\lim_{x\to 0} \gamma(x) > 0$; and (2) for any $x \leq b^*$ such that $\gamma(x) \leq 0$, we have that $\gamma'(x) \geq 0$. It then follows that $\gamma(b^*) \geq 0$. Since $\gamma(1) < 0$, there exists $x \in [b^*, 1]$ such that $\gamma(x) = 0$.

For (1),

$$\begin{aligned} \lim_{x\to 0} \gamma(x) &= 1 - \frac{1}{n-1} \lim_{x\to 0} \frac{\int_0^x y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y}{x^{\frac{n}{n-1}}} \\ &= 1 - \frac{1}{n-1} \lim_{x\to 0} \frac{x^{\frac{n}{n-1}} f(x)}{\frac{n}{n-1} x^{\frac{1}{n-1}}} \\ &= 1 - \frac{1}{n} \lim_{x\to 0} x f(x) \\ &= 1. \end{aligned}$$

For (2), for any $x \leq b^*$ and $\gamma(x) \leq 0$,

$$\begin{aligned} \gamma'(x) &= -\frac{n}{n-1} f(x) + \frac{n}{(n-1)^2} x^{-\frac{2n-1}{n-1}} \int_0^x y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y \\ &\geq -\frac{n}{n-1} f(x) + \frac{n}{n-1} \frac{1-F(x)}{x} \\ &\geq 0, \end{aligned}$$

where the first inequality follows from the definition of the function γ and the assumption that $\gamma(x) \leq 0$, and the second inequality is due to the fact that $\psi(x) \leq 0$ for $x \leq b^*$.

D Proof of Theorem 3

Recall that $\bar{b}_F$ satisfies

$$\gamma(\bar{b}_F) = 1 - F(\bar{b}_F) - \frac{1}{n-1} \bar{b}_F^{-\frac{n}{n-1}} \int_0^{\bar{b}_F} y^{\frac{n}{n-1}} f(y)\ \mathrm{d}y = 0,$$

and $G_F^*(r) = \bar{b}_F^{-\frac{1}{n-1}} r^{\frac{1}{n-1}}$ with support $[0, \bar{b}_F]$.

Proposition 3. *Suppose that there are n bidders and each bidder's valuation is distributed* *according to F where $xf(x)$ is weakly increasing in x. Then,*

$$\inf_{\pi\in\Pi} REV(G_F^*,\pi) \geq n\bar{b}_F(1-F(\bar{b}_F)).$$

Proof. We calculate the ex post revenue of the auctioneer as follows:

$$REV(G_F^*,v) = \begin{cases} \bar{b}_F^{-\frac{1}{n-1}}\left[\frac{1}{n}v(1)^{\frac{n}{n-1}} + \frac{n-1}{n}v(2)^{\frac{n}{n-1}}\right], & \text{if } v(1) \leq \bar{b}_F; \\ \frac{\bar{b}_F}{n} + \frac{n-1}{n}\bar{b}_F^{-\frac{1}{n-1}}v(2)^{\frac{n}{n-1}}, & \text{if } v(2) \leq \bar{b}_F < v(1); \\ v(2), & \text{if } v(2) > \bar{b}_F. \end{cases}$$

Let

$$u(x) = \begin{cases} \frac{1}{n}\bar{b}_F^{-\frac{1}{n-1}}x^{\frac{n}{n-1}}, & \text{if } x \leq \bar{b}_F; \\ \frac{\bar{b}_F}{n}, & \text{if } x > \bar{b}_F. \end{cases}$$

One can easily verify that

$$REV(G_F^*,v) \geq \sum_{i\in I} u(v_i)$$

for all $v \in V$. It follows that for any $\pi \in \Pi$,

$$\begin{aligned} REV(G_F^*,\pi) &\geq \int_V \sum_{i\in I} u(v_i)\, d\pi(v) \\ &= n\int_0^1 u(x)\, dF(x) \\ &= n\left[\int_0^{\bar{b}_F} \frac{1}{n}\bar{b}_F^{-\frac{1}{n-1}}x^{\frac{n}{n-1}}\, dF(x) + \frac{\bar{b}_F}{n}(1-F(\bar{b}_F))\right] \\ &= n\bar{b}_F(1-F(\bar{b}_F)), \end{aligned}$$

where the last equality follows from the construction of $\bar{b}_F$. □

Proposition 4. *Suppose that there are n bidders and each bidder's valuation is distributed according to F where $xf(x)$ is weakly increasing in x. Then, there exists a correlation structure $\pi_F^* \in \Pi$ such that*

$$\max_{M\in\hat{\mathcal{M}}} REV(M,\pi_F^*) = n\bar{b}_F(1-F(\bar{b}_F)).$$

Proof. Consider the correlation structure π_F^* induced by the following symmetric density function:

$$\eta_F^*(v) = \begin{cases} \frac{1}{(n-1)v(1)^2}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right] \\ \qquad\qquad\qquad\qquad \text{if } 0 < v(n) = v(n-1) = \ldots = v(2) \leq v(1) < \bar{b}_F, \\ \frac{1}{n-1}\frac{f(v(1))}{\bar{b}_F(1-F(\bar{b}_F))}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right] \\ \qquad\qquad\qquad\qquad \text{if } 0 < v(n) = v(n-1) = \ldots = v(2) < \bar{b}_F < v(1) < 1, \\ \times_{i\in I} f(v_i) \\ \qquad\qquad\qquad\qquad \text{if } v_i \in \{0, \bar{b}_F, 1\} \text{ for some } i, \\ 0 \\ \qquad\qquad\qquad\qquad \text{if otherwise.} \end{cases}$$

It is straightforward to verify that $\pi_F^* \in \Pi$. Note that the density function has several features. First, in the region in which $v_i = v(1)$, all the $\{v_j\}_{j\neq i}$ are maximally positively correlated. Second, in the region in which $0 < v(n) = v(n-1) = \ldots = v(2) < \bar{b}_F < v(1) < 1$, $v(1)$ and $(v(2), v(3), \ldots, v(n))$ and independently distributed. More importantly, the density function is such that the conditional virtual value of agent i in the region in which $0 < v(n) = v(n-1) = \ldots = v(2) \leq v(1) = v_i < \bar{b}_F$ is zero. This ensures that the auctioneer is indifferent over a large class of auction formats, which makes $\eta_F^*(v)$ a promising candidate for our analysis.[23]

Next, we calculate $\max_{M\in\hat{\mathcal{M}}} REV(M, \pi_F^*)$. This can be done via standard arguments (of calculating conditional virtual values and performing pointwise maximization subject to the constraint that bidders who do not have the highest value do not get the object) à la Myerson (1981).

[23]Our construction is inspired by that in Zhang (2021)—the density function $\eta_F^*(v)$ in the region $(0, \bar{b}_F)^n$ is similar to the density function in the region $(0,1)^n$ in Zhang (2021). Due to the probability mass condition imposed in Zhang (2021), the construction of the density function in his analysis is somewhat simpler than ours as there is essentially a single region. This approach of reverse engineering the worst-case scenario from manipulating the virtual values is also used in other papers, such as Roesler and Szentes (2017) and Brooks and Du (2021).

It is straightforward to calculate that the conditional virtual value of agent i is

$$
\begin{aligned}
\phi_i(v) &= v_i - \frac{1 - \int_{v(2)}^{v(1)} \frac{\frac{1}{(n-1)z^2}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]}{\frac{1}{(n-1)v(2)}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]}\,\mathrm{d}z}{\frac{\frac{1}{(n-1)v(1)^2}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]}{\frac{1}{(n-1)v(2)}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]}} \\
&= v_i - \frac{1 - \int_{v(2)}^{v(1)} \frac{v(2)}{z^2}\,\mathrm{d}z}{\frac{v(2)}{v(1)^2}} \\
&= v_i - v(1) \\
&= 0
\end{aligned}
$$

in the region in which $0 < v(n) = v(n-1) = \ldots = v(2) \le v(1) = v_i < \bar{b}_F$, and is

$$
\begin{aligned}
\phi_i(v) &= v_i - \frac{1 - \int_{v(2)}^{\bar{b}_F} \frac{v(2)}{z^2}\,\mathrm{d}z - \int_{\bar{b}_F}^{v(1)} \frac{\frac{1}{n-1}\frac{f(z)}{\bar{b}_F(1-F(\bar{b}_F))}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]}{\frac{1}{(n-1)v(2)}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]}\,\mathrm{d}z}{\frac{\frac{1}{n-1}\frac{f(v(1))}{\bar{b}_F(1-F(\bar{b}_F))}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]}{\frac{1}{(n-1)v(2)}\left[v(2)f(v(2)) - \frac{v(2)^{-\frac{n}{n-1}}}{n-1}\int_0^{v(2)} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]}} \\
&= v_i - \frac{1 - \int_{v(2)}^{\bar{b}_F} \frac{v(2)}{z^2}\,\mathrm{d}z - \frac{F(v(1)) - F(\bar{b}_F)}{\bar{b}_F(1-F(\bar{b}_F))}v(2)}{\frac{f(v(1))}{\bar{b}_F(1-F(\bar{b}_F))}v(2)} \\
&= v_i - \frac{1 - F(v(1))}{f(v(1))} \\
&\geq 0
\end{aligned}
$$

in the region in which $0 < v(n) = v(n-1) = \ldots = v(2) < \bar{b}_F < v(1) = v_i < 1$.

Therefore, under the correlation structure π^*, the optimal standard dominant-strategy mechanism generates an expected revenue of

$$
n\int_{\bar{b}_F}^{1}\int_0^{\bar{b}_F} \frac{1}{n-1}\frac{f(v_1)}{\bar{b}_F(1-F(\bar{b}_F))}\left[v_2 f(v_2) - \frac{v_2^{-\frac{n}{n-1}}}{n-1}\int_0^{v_2} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]\cdot\left(v_1 - \frac{1-F(v_1)}{f(v_1)}\right)\mathrm{d}v_2\,\mathrm{d}v_1
$$

$$
\begin{aligned}
=&\frac{n}{n-1}\frac{1}{\bar{b}_F(1-F(\bar{b}_F))}\int_{\bar{b}_F}^{1}(v_1 f(v_1)-1+F(v_1))\int_0^{\bar{b}_F}\left[v_2 f(v_2)-\frac{v_2^{-\frac{n}{n-1}}}{n-1}\int_0^{v_2} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\right]\mathrm{d}v_2\,\mathrm{d}v_1\\
=&\frac{n}{n-1}\frac{1}{\bar{b}_F(1-F(\bar{b}_F))}\int_{\bar{b}_F}^{1}(v_1 f(v_1)-1+F(v_1))\int_0^{\bar{b}_F}\bar{b}_F^{-\frac{1}{n-1}} y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y\,\mathrm{d}v_1\\
=&\frac{n}{n-1}\frac{1}{\bar{b}_F(1-F(\bar{b}_F))}\int_{\bar{b}_F}^{1}(v_1 f(v_1)-1+F(v_1))(n-1)\bar{b}_F(1-F(\bar{b}_F))\,\mathrm{d}v_1\\
=&n\int_{\bar{b}_F}^{1}(v_1 f(v_1)-1+F(v_1))\,\mathrm{d}v_1\\
=&n\bar{b}_F(1-F(\bar{b}_F)),
\end{aligned}
$$

where the third line changes the order of integration and the fourth line uses the equation $\gamma(\bar{b}_F)=0$. □

Theorem 3 follows from Proposition 3 and Proposition 4.

In its essentials, the above show that (G_F^*, π_F^*) is a saddle point in this problem. To pin down this saddle point, we work with a different correlation strcuture (from the one in Section 5). In Section 5, we identified the worst-case corrlation structure $\pi^{r_n^*}$ for the robustly optimal reserve price r_n^*, but r_n^* does not generate the highest revenue against $\pi^{r_n^*}$ among all standard dominant-strategy mechanisms (for instance, the second price auction with the reserve price $c_n(r_n^*)$ generates a strictly higher expected revenue than r_n^* does against $\pi^{r_n^*}$).[24] Having said that, $\pi^{r_n^*}$ and π_F^* both converge to the maximally positive correlation when n grows large.

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[24] To see that $c_n(r_n^*)$ generates a strictly higher expected revenue than r_n^* does against the correlation structure $\pi^{r_n^*}$, note that

1. $\pi^{r_n^*}$ only places positive probability in the regions $V^{r_n^*,\emptyset}$ and $V^{r_n^*,I}$,
2. for any $v \in V^{r_n^*,\emptyset}$, the ex post revenue for both mechanisms are 0, and
3. for $\pi^{r_n^*}$-almost all v in the region $V^{r_n^*,I}$, $c_n(r_n^*)$ generates a strictly higher ex post revenue than r_n^* does (since $v(1) > c_n(r_n^*) > v(2) > r_n^*$).

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