Order Statistics of Large Samples: Theory and an Application to Robust Auction Design*

Wei He† Jiangtao Li‡ Weijie Zhong§

August 6, 2022

Abstract

We prove an elementary property of order statistics that bounds the k-th largest order statistic of a given sample using the largest order statistic of a (random) subsample. This property is applied to the design of combinatorial auctions when the auctioneer has limited statistical information about the joint distribution of the bidders' valuations. The VCG mechanism is asymptotically optimal—its revenue-guarantee differs from the highest revenue-guarantee by at most $O(\frac{1}{n})$.

*We thank Satoru Takahashi for helpful discussions and Kexin Wang for excellent research assistance.

†Department of Economics, The Chinese University of Hong Kong, hewei@cuhk.edu.hk

‡School of Economics, Singapore Management University, jtli@smu.edu.sg

§Graduate School of Business, Stanford University, weijie.zhong@stanford.edu

1 Introduction

The theory of order statistics is an integral part of probability theory and a requisite component in the toolbox of researchers in various fields including economics. Applications in economics include modeling auctions, optimizing production processes, estimating parameters of a distribution, among many others. For a notable example, results on order statistics are the basis of much of the auction literature—for the single-unit auction, the full surplus is the expectation of the highest value, the expected revenue from the first-price auction with no reserve price is the expectation of the highest bid, and the expected revenue from the second-price auction with no reserve price is the expectation of the second highest value.

In this paper, we prove an elementary property of order statistics that bounds the k-th largest order statistic of a given sample using the largest order statistic of a (random) subsample. Among other things, this implies that in a variety of settings, the worst-case expectations of order statistics are asymptotically equivalent. We apply this property to the design of combinatorial auctions when the auctioneer has limited statistical information about the joint distribution of the bidders' valuations. By bounding the expected revenue from the VCG mechanism (see Vickrey (1961), Clarke (1971), and Groves (1973)) from below using a few properly constructed order statistics, we show that the VCG mechanism is asymptotically revenue-maximizing.

Traditional models of optimal auction design pin down relatively simple auction formats that maximize revenue under strong assumptions about the environment. For instance, the celebrated work of Myerson (1981) works with the allocation of a single unit of an object and makes the explicit assumption of independent values, as well as implicit assumptions that both the auctioneer and the bidders are fully Bayesian with an accurate common prior belief. In practice, however, these assumptions are rarely satisfied. For example, two of the most profitable auctions, the Google Ads Auction and the Meta Ads Auction,[1] are combinatorial and involve highly correlated valuations. Given the frequency and scale of these ads auctions, the data accessibility and computation capability constraints prohibit the auctioneer to even form an accurate belief in real-time, not to say solving the revenue maximization problem. Our result suggests that the co-existence of all these nuances is in fact a blessing instead of a curse: the scale of the auction

[1] In 2021, Google and Meta (formerly Facebook Inc) generated ads revenue of 209.49 billion and 114.93 billion U.S. dollars, respectively. Google adopts generalized second price (GSP) auction and Meta adopts VCG auction.

and the lack of knowledge lead to the robust optimality of the VCG mechanism.

We consider a combinatorial auction, where an auctioneer sells a set K of heterogeneous objects to n bidders. Each bidder i has her own private valuation function $v_i : 2^K \to \mathbb{R}_+$ that specifies bidder i's valuation for each possible subset of objects. We assume that the possible valuation functions are uniformly bounded and exhibit no complementarity. The auctioneer has limited statistic information about the joint distribution of the bidders' valuation. In the baseline model, the auctioneer only knows that the marginal distribution of each bidder's private type lies in an arbitrary *ambiguity set* $\Pi \subseteq \Delta V$. All joint distributions that are compatible with the ambiguity set are considered plausible.

The auctioneer evaluates mechanisms in terms of *revenue-guarantee*—the worst-case expected revenue where the worst-case is taken over all joint distributions that are considered plausible. We show that the VCG mechanism is asymptotically optimal—its revenue-guarantee differs from the highest revenue-guarantee by at most $O(\frac{1}{n})$. The following example illustrates the intuition of this result in a simple setting.

Example 1 (Single-unit auction)**.** Consider a single-unit auction with n bidders, where $\Pi = \{U[0,1]\}$. It is well-known that when the bidders' values are independent, the expectation of the second highest value is $\frac{n-1}{n+1}$, which approaches $\frac{n}{n+1}$, the expectation of the highest value as n gets large. This seems intuitive as the second highest value is a good approximation of the highest value when values are independent.[2]

Perhaps surprisingly, this is not the case if the values are correlated. For a simple example, consider the joint distribution defined by randomly drawing $q \sim U[0,1]$ and taking $v_1 = q$ and $v_i = 1 - q$ for all $i \neq 1$. Obviously, the distributions of the highest and the second highest values do not depend on n, and it is easy to calculate that $\mathbb{E}[v^{(1)}] = \frac{3}{4}$ and $\mathbb{E}[v^{(2)}] = \frac{1}{2}$.[3] Thus, for some correlation structures, the second highest value could be a poor approximation of the highest value.[4]

Our key observation is that the second highest value remains a good approximation of the highest value under the worst-case correlation structure. $\mathbb{E}[v^{(1)}]$ is minimized when all values are maximally positively correlated, and it is simply the expected value of one randomly selected bidder. We now compare $\mathbb{E}[v^{(2)}]$ with the expected value of a randomly selected bidder (without specifying the correlation

[2] Bulow and Klemperer (1996) shows that adding one more bidder is sufficient for the second highest value to outperform the optimal mechanism for any finite n.

[3] We use $v^{(k)}$ to denote the k-th largest order statistic in $\{v_1, v_2, \ldots, v_n\}$.

[4] It is obvious that for the proposed distributions, adding extra bidders has no use at all.

structure). Let $\boldsymbol{i}$ be a uniform randomization of bidder identities. We have

$$\begin{aligned}
\mathbb{E}[v^{(2)}] =& \mathbb{E}[v^{(2)}|v^{(2)} \geq v_{\boldsymbol{i}}]\mathbb{P}(v^{(2)} \geq v_{\boldsymbol{i}}) + \mathbb{E}[v^{(2)}|v^{(2)} < v_{\boldsymbol{i}}]\mathbb{P}(v^{(2)} < v_{\boldsymbol{i}}) \\
\geq& \mathbb{E}[v_{\boldsymbol{i}}|v^{(2)} \geq v_{\boldsymbol{i}}]\mathbb{P}(v^{(2)} \geq v_{\boldsymbol{i}}) \\
=& \mathbb{E}[v_{\boldsymbol{i}}] - \mathbb{E}[v_{\boldsymbol{i}}|v^{(2)} < v_{\boldsymbol{i}}]\mathbb{P}(v^{(2)} < v_{\boldsymbol{i}}) \\
\geq& \mathbb{E}[v_{\boldsymbol{i}}] - \frac{1}{n}.
\end{aligned}$$

The two equalities apply the law of iterated expectations. The second inequality is because, for $v^{(2)} < v_{\boldsymbol{i}}$, $\boldsymbol{i}$ must be the bidder with the highest value, which happens with probability $\frac{1}{n}$. Therefore, the worst-case expectation of the second highest value differs from that of the largest value by at most $\frac{1}{n}$. The second-price auction with no reservation price is asymptotically optimal in large markets.

The methodology in Example 1 is generalized to show the main result. We show that (1) the expected revenue from the VCG mechanism can be bounded from below using a few properly constructed order statistics and (2) the highest revenue-guarantee can be bounded from above by the total welfare under the efficient allocation when the bidders' valuations are maximally positively correlated. Similar analysis as in Example 1 shows that these two bounds differ by $O(\frac{1}{n})$.

Appendix A extends our analysis to the case of asymmetric bidders. The key assumption on the ambiguity set is what we call a *subsample sufficiency* property. In words, it says that for any object, the worst-case expectation of the highest individual utility for the object is approximately the same in the full sample and in any subsample that is not "too small." Theorem 3 extends the asymptotic optimality of the VCG mechanism to settings with asymmetric ambiguity set and restricted correlation structures.

2 Order statistics of large samples: theory

In this section, we prove an elementary property of order statistics that bounds the k-th largest order statistic of a given sample using the largest order statistic of a (random) subsample.

Let $M = \{1, 2, \ldots, m\}$, $m \geq 2$. Denote $\{X_i\}_{i=1}^m$ as an m-dimensional random variable, where all X_i's are uniformly bounded in the interval $[\underline{x}, \bar{x}]$. We write $F \in \Delta[\underline{x}, \bar{x}]^m$ to denote the joint distribution of $\{X_i\}_{i=1}^m$. For any $I \subseteq M$, let $X_I^{(k)}$ denote the k-th largest order statistic of $\{X_i\}_{i \in I}$. For any $r \leq m$, let $\boldsymbol{I}(r)$ denote the uniform random sample of r distinct elements from M.

Lemma 1. *For any F, $k \leq m$, and $r \leq m-k+1$, we have*

$$\mathbb{E}_F\left[X_M^{(k)}\right] \geq \mathbb{E}_F\left[X_{\boldsymbol{I}(r)}^{(1)}\right] - (\bar{x} - \underline{x})\left(1 - \frac{\binom{m-k+1}{r}}{\binom{m}{r}}\right).$$

Proof.

$$\begin{aligned}
&\mathbb{E}_F\left[X_M^{(k)}\right] \\
=&\mathbb{E}_F\left[X_M^{(k)} | X_M^{(k)} \geq X_{\boldsymbol{I}(r)}^{(1)}\right]\mathbb{P}\left(X_M^{(k)} \geq X_{\boldsymbol{I}(r)}^{(1)}\right) + \mathbb{E}_F\left[X_M^{(k)} | X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}\right]\mathbb{P}\left(X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}\right) \\
\geq&\mathbb{E}_F\left[X_{\boldsymbol{I}(r)}^{(1)} | X_M^{(k)} \geq X_{\boldsymbol{I}(r)}^{(1)}\right]\mathbb{P}\left(X_M^{(k)} \geq X_{\boldsymbol{I}(r)}^{(1)}\right) + \underline{x}\mathbb{P}\left(X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}\right) \\
=&\mathbb{E}_F\left[X_{\boldsymbol{I}(r)}^{(1)}\right] - \mathbb{E}_F\left[X_{\boldsymbol{I}(r)}^{(1)} | X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}\right]\mathbb{P}\left(X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}\right) + \underline{x}\mathbb{P}\left(X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}\right) \\
\geq&\mathbb{E}_F\left[X_{\boldsymbol{I}(r)}^{(1)}\right] - (\bar{x} - \underline{x})\mathbb{P}\left(X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}\right),
\end{aligned} \tag{1}$$

where the second and the fourth line apply the law of iterated expectations, and the third and the fifth line follow from the assumption that all X_i's are uniformly bounded in the interval $[\underline{x}, \bar{x}]$.

When $X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}$, $\boldsymbol{I}(r)$ necessarily contains at least one index i such that $X_i > X_M^{(k)}$. Note that there are at most $k-1$ such indices. Since $\boldsymbol{I}(r)$ is the uniform random sample of r distinct elements from M, the probability that at least one of $k-1$ specific indices is contained in $\boldsymbol{I}(r)$ is $1 - \frac{\binom{m-k+1}{r}}{\binom{m}{r}}$. Thus,

$$\mathbb{P}\left(X_M^{(k)} < X_{\boldsymbol{I}(r)}^{(1)}\right) \leq 1 - \frac{\binom{m-k+1}{r}}{\binom{m}{r}}. \tag{2}$$

Lemma 1 follows from Equations (1) and (2). *Q.E.D.*

Lemma 1 holds for any number of random variables m and any joint distribution F. We now apply Lemma 1 to the case in which the number of random variables is large and there is limited statistical information about the joint distribution. In this setting, we establish an asymptotic equivalence result on the worst-case expectations of order statistics. Let $\{X_i\}_{i=1}^{\infty}$ be an infinite sequence of random variables. As before, we assume that all X_i's are uniformly bounded in the interval $[\underline{x}, \bar{x}]$. Let $\mathcal{F}$ denote the collection of joint distributions the modeler perceives plausible.

Definition 1 (Subsample sufficiency)**.** *For an increasing sequence $r(n)$ and a* *decreasing sequence* $\chi(n)$, $\mathcal{F}$ *is* r-***subsample*** χ-***sufficient*** *if for all* n,

$$\inf_{F\in\mathcal{F}} \mathbb{E}_F\left[X^{(1)}_{\boldsymbol{I}(r(n))}\right] \geq \inf_{F\in\mathcal{F}} \mathbb{E}_F\left[X^{(1)}_{[n]}\right] - \chi(n).^{5}$$

In words, Definition 1 says that to achieve the worst-case expectation of the largest order statistic up to a $\chi(n)$ difference, it is not necessary to sample all X_i's. Instead, it suffices to consider a random size-$r(n)$ subset of random variables. When the family of distributions $\mathcal{F}$ is symmetric and exhibits full ambiguity in the correlation structure (for example, when $\mathcal{F}$ is the collection of all joint distributions that have the same marginal distribution on all dimensions), it is 1-subsample 0-sufficient.[6]

Theorem 1 below bounds the difference between the worst-case expectations of order statistics.

Theorem 1. *If* $\mathcal{F}$ *is* r*-subsample* χ*-sufficient, then for any* k,

$$\inf_{F\in\mathcal{F}} \mathbb{E}_F\left[X^{(1)}_{[n]}\right] - \inf_{F\in\mathcal{F}} \mathbb{E}_F\left[X^{(k)}_{[n]}\right] \leq O\left(\frac{r(n)}{n} + \chi(n)\right).$$

Proof.

$$\begin{aligned}
\inf_{F\in\mathcal{F}} \mathbb{E}_F\left[X^{(k)}_{[n]}\right] &\geq \inf_{F\in\mathcal{F}}\left[\mathbb{E}_F\left[X^{(1)}_{\boldsymbol{I}(r(n))}\right] - (\bar{x}-\underline{x})\left(1 - \frac{\binom{n-k+1}{r(n)}}{\binom{n}{r(n)}}\right)\right] \\
&= \inf_{F\in\mathcal{F}} \mathbb{E}_F\left[X^{(1)}_{\boldsymbol{I}(r(n))}\right] - O\left(\frac{r(n)}{n}\right) \\
&\geq \inf_{F\in\mathcal{F}} \mathbb{E}_F\left[X^{(1)}_{[n]}\right] - O\left(\frac{r(n)}{n} + \chi(n)\right),
\end{aligned}$$

where the first inequality follows from Lemma 1 and the last inequality follows from Definition 1. *Q.E.D.*

Note that there is a trade-off between r and χ: to achieve small approximation error χ, one necessarily needs to choose a large subsample.[7] Theorem 1 states that if a diminishing approximation error can be achieved by considering a subset of size $o(n)$, then the worst case expectation of all finite order statistics are asymptotically the same. In Section 3, we utilize Theorem 1 to study robustly optimal auction design under symmetric $\mathcal{F}$ that exhibits full ambiguity in the correlation

[5] $[n] = \{1, 2, \ldots, n\}$ for all n.

[6] We say that a family of distributions $\mathcal{F}$ is symmetric if $F_\tau \in \mathcal{F}$ for any $F \in \mathcal{F}$ and any permutation $\tau : \mathbb{N}_+ \to \mathbb{N}_+$, where F_τ is the joint distribution of $\{X_{\tau(1)}, X_{\tau(2)}, \ldots\}$.

[7] Any $\mathcal{F}$ is trivially n-subsample 0-sufficient and 1-subsample $(\bar{x}-\underline{x})$-sufficient. However, these pairs of r, χ do not provide a meaningful bound in Theorem 1.

structure, which satisfies 1-subsample 0-sufficiency. The extension to general $\mathcal{F}$ is straightforward and is discussed in Appendix A.

3 The combinatorial auction

In this section, we consider a combinatorial auction of heterogeneous objects. Let $K = \{1, 2, \dots, k\}$ be the finite set of objects for sale, and $N = \{1, 2, \dots, n\}$ the set of potential buyers. As we eventually focus on large markets, we assume that $n \geq k+1$. Each bidder i has her own private valuation function $v_i : 2^K \to \mathbb{R}_+$ that specifies bidder i's valuation for each possible subset of objects: for a subset of objects $A \subseteq K$, $v_i(A)$ is bidder i's valuation of the bundle A. We assume that valuations are monotonically non-decreasing ($v_i(A) \leq v_i(B)$ whenever $A \subseteq B$), normalized so that $v_i(\emptyset) = 0$, and scaled to lie in $[0, 1]$. Each bidder i knows her own valuation function v_i, and this is common knowledge. We also refer to v_i as bidder i's type. For ease of exposition, in the baseline model, we assume that all bidders are ex ante identical. We denote by V the set of possible valuation functions of each bidder i, where V is a complete and separable metric space. (In Appendix A, we extend our analysis to the case of asymmetric bidders).

We assume that the bidders' valuation functions satisfy the following standard property:

Assumption 1 (Complement-free). *v_i is complement-free.*[8] *That is, for any $A, B \subseteq K$,*

$$v_i(A) + v_i(B) \geq v_i(A \cup B).$$

For any $F \in \Delta V^n$, let $\Phi_i(F)$ denote the marginal distribution of F on the i-th dimension. The auctioneer lacks information about the joint distribution of the bidder's types, and believes any joint distribution in $\mathcal{F}$ is a plausible candidate, where

$$\mathcal{F} = \left\{ F \in \Delta V^n \,\middle|\, \Phi_i(F) \in \Pi \text{ for all } i \right\}$$

for some exogenously given $\Pi \subseteq \Delta V$.

[8] Submodular valuation functions have received much attention in combinatorial auctions; v_i is submodular if for any pair of nested subsets of objects $A \subseteq B \subseteq K$ and any single object x,

$$v_i(A \cup \{x\}) - v_i(A) \geq v_i(B \cup \{x\}) - v_i(B).$$

Our assumption of complement-free valuation function is a strictly weaker notion than submodularity.

Remark. We do not place any restriction on the collection of distributions $\Pi \subseteq \Delta V$. This flexibility means that we can apply our results to various settings in which Π is constructed to reflect additional natural characteristics. For example, by modeling Π to be a singleton set, we could accommodate the correlation-robust auction model; see Bei et al. (2019), He and Li (2022), and Zhang (2021). By modeling Π to be the collection of distributions that satisfy certain moment conditions, we could accommodate the analysis in Che (2020) and Suzdaltsev (2020). These features are consistent with our framework, but are not required for our main results.

Mechanisms/ payoff functions

An allocation is a vector of sets $A = (A_1, A_2, \ldots, A_n)$, $A_i \cap A_j = \emptyset$, where A_i denotes the bundle allocated to bidder i (it is not required that all items are allocated).[9] Let $\mathcal{A}_n$ denote the set of all allocations with n bidders. Let $A^*(v) \in \arg\max_{A \in \mathcal{A}_n} \sum_i v_i(A_i)$ denote an efficient allocation under v.

Rather than defining the set of feasible mechanisms explicitly, it suffices for our purpose to directly define the set of feasible payoff functions that are implementable.

Definition 2. *We define the* VCG payoff t_{vcg} *as a function of the type profile as follows: for each* $v \in V^n$*,*

$$t_{vcg}(v) = \sum_{i=1}^{n} \left(\sup_{A \in \mathcal{A}_{n-1}} \sum_{j \neq i} v_j(A_j) - \sum_{j \neq i} v_j(A^*_j(v)) \right).$$

That is, t_{vcg} is the payoff function implemented through the VCG mechanism.

Let $\mathcal{T}$ denote the set of "feasible payoffs" functions that are implementable (without fully specifying it). We assume that (1) the VCG mechanism is feasible and (2) any feasible mechanism satisfies the individual rationality requirement—the payoff of each bidder i is nonnegative. That is, (1) $t_{vcg} \in \mathcal{T}$ and (2) $t(v) \leq \sup_{A \in \mathcal{A}_n} \sum_i v_i(A_i)$ for any $t \in \mathcal{T}$.

Revenue-guarantee

For any $t \in \mathcal{T}$ and $F \in \mathcal{F}$, we denote by $R(t, F) = \int_V t(v)F(\mathrm{d}v)$ the auctioneer's expected revenue when the mechanism uses the payoff function t and

[9] For notational simplicity, we only consider deterministic allocations. Given the concerns of this paper, allowing for randomized allocations would not affect what follows.

the joint distribution is F. We define the revenue-guarantee of a mechanism t as $\inf_{F\in\mathcal{F}} R(t,F)$. The auctioneer's objective is to choose a mechanism that generates the highest revenue-guarantee:

$$\sup_{t\in\mathcal{T}}\inf_{F\in\mathcal{F}} R(t,F).$$

Asymptotic optimality of the VCG mechanism

Theorem 2. *Under Assumption 1*

$$\inf_{F\in\mathcal{F}} R(t_{vcg},F)\geq \sup_{t\in\mathcal{T}}\inf_{F\in\mathcal{F}} R(t,F)-O\left(\frac{1}{n}\right).$$

In words, Theorem 2 says that the revenue-guarantee of the VCG mechanism differs from the highest revenue-guarantee by at most $O(\frac{1}{n})$, and hence the VCG mechanism is asymptotically optimal (among all individually rational mechanisms). The proof proceeds by bounding the expected revenue of the VCG mechanism by a few properly constructed order statistics, and then applying Lemma 1 (Theorem 1 in the case of asymmetric bidders).

Proof. Step 1. We first construct an upper bound of the highest revenue-guarantee. For any $\pi\in\Pi$, consider the maximally positive correlation F_π that puts probability one on the line segment $v_1=v_2=\ldots=v_n$. Clearly, $F_\pi\in\mathcal{F}$ for all $\pi\in\Pi$. Almost surely under the joint distribution F_π, for any $A\in\mathcal{A}_n$,

$$\begin{aligned}\sum_{i=1}^n v_i(A_i)\leq&\sum_{i=1}^n\sum_{o\in A_i} v_i(\{o\})\\ =&\sum_{i=1}^n\sum_{o\in A_i} v_1(\{o\})\\ \leq&\sum_{o=1}^k v_1(\{o\}),\end{aligned}$$

where the first inequality follows from Assumption 1, the first equality is because for any v in the support of F_π, $v_i=v_1$ for all i, and the second inequality follows from a reordering of the terms in the summation. Therefore,

$$\begin{aligned}\sup_{t\in\mathcal{T}}\inf_{F\in\mathcal{F}} R(t,F)\leq&\sup_{t\in\mathcal{T}}\inf_{\pi\in\Pi} R(t,F_\pi)\\ \leq&\inf_{\pi\in\Pi}\sup_{t\in\mathcal{T}} R(t,F_\pi)\end{aligned}$$

$$\leq \inf_{\pi \in \Pi} \mathbb{E}_{F_\pi} \left[\sum_{o=1}^{k} v_1(\{o\}) \right]. \tag{3}$$

Step 2. We proceed to establish a lower bound of the revenue-guarantee of the VCG mechanism by constructing, for each i, an allocation $A^i \in \mathcal{A}_{n-1}$ of the objects to the bidders other than bidder i. Clearly, for any such profile (A^i),

$$\begin{aligned} t_{vcg}(v) &= \sum_{i=1}^{n} \left(\sup_{A \in \mathcal{A}_{n-1}} \sum_{j \neq i} v_j(A_j) - \sum_{j \neq i} v_j(A_j^*(v)) \right) \\ &\geq \sum_{i=1}^{n} \left(\sum_{j \neq i} v_j(A_j^i) - \sum_{j \neq i} v_j(A_j^*(v)) \right) \end{aligned} \tag{4}$$

For each i, we construct allocation $A^i \in \mathcal{A}_{n-1}$ via the following algorithm:

Algorithm. Set $A_j^i = \emptyset$ for all j. Set $O = A_i^*$.

(1). For each $j \neq i$:

If $A_j^* \neq \emptyset$, set $A_j^i = A_j^*$.

Let $\bar{N} = \{j \in N : A_j^i = \emptyset, j \neq i\}$.

(2). If $O \neq \emptyset$, pick $o \in O$.

Set $A_j^i = \{o\}$ for some $j \in \arg\max_{j' \in \bar{N}} v_{j'}(\{o\})$.

Update $O \leftarrow O \setminus \{o\}$ and $\bar{N} \leftarrow \bar{N} \setminus \{j\}$.

(3). Repeat (2) until $O = \emptyset$.

(4). Return allocation $A^i = (A_1^i, A_2^i, \ldots, A_{i-1}^i, A_{i+1}^i, \ldots, A_n^i)$.

In words, if an object is allocated to a bidder other than bidder i under A^*, the object is still allocated to that bidder. We then iteratively pick an object o that is allocated to bidder i under A^*, and allocate the object to the bidder j whose value for the object $v_j(\{o\})$ is the highest among all the bidders who are not allocated any object yet. For each $o \in A_i^*$, define j_o to be the index j such that $A_j^i = \{o\}$.

It follows from Equation (4) that

$$\begin{aligned} t_{vcg}(v) &\geq \sum_{i=1}^{n} \left(\sum_{j \neq i} v_j(A_j^i) - \sum_{j \neq i} v_j(A_j^*(v)) \right) \\ &= \sum_{i=1}^{n} \sum_{o \in A_i^*} v_{j_o}(\{o\}) \end{aligned}$$

$$\geq \sum_{o=1}^{k} \max_j{}^{(k+1)} v_j(\{o\}), \tag{5}$$

where $\max_j^{(k+1)}$ denotes the $(k+1)$-th largest element when j traverses $\{1, \ldots, n\}$. The second inequality follows from the construction of A^i: when an object $o \in A_i^*$ is being allocated, it is allocated to the bidder j whose value for the object $v_j(\{o\})$ is the highest among all the bidders who are not allocated any object yet. Since each iteration assigns at least one good to one bidder and there are at most k goods, we have $v_{j_o}(\{o\})$ must be at least the $(k+1)$-th highest value among all $v_i(\{o\})$.

Equation (5) then implies that for any $F \in \mathcal{F}$,

$$\begin{aligned} R(t_{vcg}, F) \geq & \mathbb{E}_F \left[\sum_{o=1}^{k} \max_j{}^{(k+1)} v_j(\{o\}) \right] \\ = & \sum_{o=1}^{k} \mathbb{E}_F \left[\max_j{}^{(k+1)} v_j(\{o\}) \right] \\ \geq & \sum_{o=1}^{k} \left(\mathbb{E}_F \left[v_1(\{o\}) \right] - \frac{k}{n} \right), \\ \implies \inf_{F \in \mathcal{F}} R(t_{vcg}, F) \geq & \inf_{\pi \in \Pi} \mathbb{E}_{F_\pi} \left[\sum_{o=1}^{k} v_1(\{o\}) \right] - \frac{k^2}{n}. \end{aligned} \tag{6}$$

The second inequality follows from Lemma 1 and F having symmetric marginals. It follows from Equations (3) and (6) that

$$\inf_{F \in \mathcal{F}} R(t_{vcg}, F) \geq \sup_{t \in \mathcal{T}} \inf_{F \in \mathcal{F}} R(t, F) - O\left(\frac{1}{n}\right).$$

Q.E.D.

4 Related literature

This paper joins the burgeoning literature of robust mechanism design; see Bergemann and Morris (2012) and Carroll (2019) for recent surveys on robust mechanism design and references therein. An important strand of this literature considers settings in which the designer does not have reliable information about the agents' hierarchies of beliefs while assuming the knowledge of the payoff environment; see, for example, Bergemann and Morris (2005), Chung and Ely (2007), Bergemann et al. (2016, 2017, 2019), Chen and Li (2018), Du (2018), Brooks and Du (2021), and Yamashita and Zhu (2020). The focus of our paper is on the uncertainty about the payoff environment.

Several recent papers have analyzed single-unit auction settings in which the auctioneer has limited statistical information of the payoff environment.[10] Bei et al. (2019), He and Li (2022), and Zhang (2021) consider an auctioneer who only has knowledge of the marginal distribution, but not the correlation structure. Neeman (2003), Koçyiğit et al. (2020), Suzdaltsev (2020), and Che (2020) consider an auctioneer who only has knowledge about certain moment conditions of the marginal distribution. There are several differences between our work and the above-mentioned papers. First, while these papers consider the single-unit auction setting, we focus on combinatorial auctions (which cover the single-unit auction as a special case). Second, rather than considering a specific set of possible joint distributions, our formulation of uncertainty set is general enough to cover many different specifications; see Section 3. Third, to establish the asymptotic optimality of the VCG mechanism in combinatorial auctions, we uncover some elementary properties on the order statistics of large samples. Lemma 1 and Theorem 1 can be readily used to show that, in the environments considered in these papers, the second-price auction with no reserve price is asymptotically optimal.

5 Conclusion

In many realistic settings, the designer might not have a complete understanding of the payoff environment and faces some uncertainty about the joint distribution of the agents' payoff types. This paper uncovers an elementary property of order statistics that bounds the k-th largest order statistic of a given sample using the largest order statistic of a (random) subsample. By establishing a lower bound of the performance of the VCG mechanism using properly constructed order statistics, we apply this property to establish the asymptotic optimality of the VCG mechanism in combinatorial auction settings. Such property of order statistics could potentially be useful for other (robust) design settings.

[10] Also see Carroll (2017) and Che and Zhong (2021) who consider a multi-dimensional screening setting in which the seller faces uncertainty about the payoff environment.

A General ambiguity set

We now extend our analysis to an arbitrary ambiguity set. We first adapt the subsample sufficiency property (Definition 1) to the combinatorial auction setting.

Definition 3 (Subsample sufficiency)**.** *For an increasing sequence $r(n)$ and a decreasing sequence $\chi(n)$, $\mathcal{F}$ is r-**subsample** χ-**sufficient** if for all n, for all $o \in K$,*

$$\inf_{F\in\mathcal{F}} \mathbb{E}_F\left[\max_{i\in\boldsymbol{I}(r(n))} v_i(\{o\})\right] \geq \inf_{F\in\mathcal{F}} \mathbb{E}_F\left[\max_{i\in[n]} v_i(\{o\})\right] - \chi(n).$$

In words, Definition 3 says that for any object, considering a random size-$r(n)$ subset of bidders, the worst case expectation of the highest individual utility for the object is at most $\chi(n)$ from that of the full set of bidders.

Theorem 3. *If Assumption 1 is satisfied and $\mathcal{F}$ is r-subsample χ-sufficient, then*

$$\inf_{F\in\mathcal{F}} R(t_{vcg}, F) \geq \sup_{t\in\mathcal{T}} \inf_{F\in\mathcal{F}} R(t, F) - O\left(\frac{r(n)}{n} + \chi(n)\right).$$

Proof. The proof is analogous to the proof of Theorem 2. Here, we only note down the key changes.

Step 1. Equation (3) is modified to

$$\sup_{t\in\mathcal{T}} \inf_{F\in\mathcal{F}} R(t, F) \leq \inf_{F\in\mathcal{F}} \mathbb{E}_F\left[\sum_{o=1}^{k} \sup_{i\in[n]} v_i(\{o\})\right] \tag{7}$$

Step 2. Equation (5) still holds in the current setting, as the derivation of Equation (5) does not depend on the distribution of bidders' valuations. Therefore,

$$\begin{aligned}
\inf_{F\in\mathcal{F}} R(t_{vcg}, F) &\geq \inf_{F\in\mathcal{F}} \sum_{o=1}^{k} \mathbb{E}_F\left[\max_{j\in[n]}{}^{(k+1)} v_j(\{o\})\right] \\
&\geq \inf_{F\in\mathcal{F}} \sum_{o=1}^{k} \mathbb{E}_F\left[\max_{j\in[n]}{}^{(1)} v_j(\{o\})\right] - O\left(\frac{r(n)}{n} + \chi(n)\right) \\
&\geq \sup_{t\in\mathcal{T}} \inf_{F\in\mathcal{F}} R(t, F) - O\left(\frac{r(n)}{n} + \chi(n)\right),
\end{aligned}$$

where the first line follows from Equation (5), the second line follows from Theorem 1, and the last line follows from Equation (7). *Q.E.D.*

Theorem 3 extends Theorem 2 by allowing the ambiguity of the correlation structure to be less extreme or the ambiguity set to be asymmetric. To illustrate Theorem 3, consider the following examples.

Example 2. The setting is the same as Example 1, except that the ambiguity set $\mathcal{F}$ exhibits limited correlation:

$$\mathcal{F} = \left\{ F \in \Delta[0,1]^n \middle| \Phi_i(F) \sim U[0,1] \text{ and } z \leq \frac{\mathbb{P}(X_i|X_{-i})}{\mathbb{P}(X_i|X'_{-i})} \leq \frac{1}{z}, \forall i, X_i, X_{-i}, X'_{-i} \right\},$$

where $z \in (0,1]$. The ambiguity set $\mathcal{F}$ satisfies "z-independence" (introduced by Cripps and Swinkels (2006)). When $z = 1$, $\mathcal{F}$ contains a unique i.i.d. $U[0,1]$ distribution. When $z \to 0$, $\mathcal{F}$ exhibits full ambiguity on correlation. Intuitively, the parameter z captures the level of dependence among bidders. Next, we show that the second price auction is asymptotically optimal under the z-independence condition.

The definition of $\mathcal{F}$ implies that $\forall F \in \mathcal{F}$, $\forall i$, $\forall X_{-i}$:

$$\mathbb{P}(v_i \in [x,1]|X_{-i}) \geq z \cdot (1-x),$$

where the RHS is the CDF of $U[1-1/z,1]$. Let $\tilde{\boldsymbol{v}} = U[1-1/z,1]$. Then, $\forall i$, $\forall X_{-i}$,

$$\tilde{\boldsymbol{v}} \leq_{FOSD} v_i \big| X_{-i}.$$

Let $\{\tilde{v}_i\}_{i\in[n]}$ be i.i.d. copies of $\tilde{\boldsymbol{v}}$. Therefore,

$$\begin{aligned}
\mathbb{E}[\max_{i\in \boldsymbol{I}(m)} v_i] \geq& \mathbb{E}[\max_{i\in \boldsymbol{I}(m)} \tilde{v}_i] \\
=& \mathbb{E}[\max_{i\in[m]} \tilde{v}_i] \\
=& 1 - \frac{1}{z}\frac{1}{1+m} \\
\geq& \mathbb{E}[\max_{i\in[n]} v_i] - \frac{1}{z}\frac{1}{1+m}.
\end{aligned} \tag{8}$$

The first inequality is by replacing the conditional distribution of each v_i with $\tilde{v}_i$. The first equality is from $\tilde{v}_i$ being i.i.d. The second equality is from calculating the first order statistic of uniform i.i.d. distribution. Note that Equation (8) implies that $\mathcal{F}$ is m-subsample $\frac{1}{z}\frac{1}{1+m}$-sufficient. When m is chosen properly (e.g. $m = \sqrt{n}$), Theorem 3 implies that the second price auction is asymptotically optimal under ambiguity set $\mathcal{F}$.

Example 3. Consider a single-unit auction setting. There exists $G \in \Delta[0,1]$ and

$\Pi \subset \Delta[0,1]$ such that $\forall G' \in \Pi$, $G \geq_{FOSD} G'$. There exists an increasing sequence $r(n)$ such that

$$\mathcal{F} = \{F \in \Delta[0,1]^n | \Phi_i(F) \in \Pi \text{ for } r(n) \text{ } i\text{'s and } \Phi_i(F) = G \text{ for remaining } i\text{'s}\}.$$

In words, it is known to the auctioneer that $r(n)$ bidders are ex ante "inferior" to the rest of the bidders. It is easy to verify that the ambiguity set $\mathcal{F}$ is $r(n)+1$-subsample 0-sufficient, because sampling any $r(n)+1$ bidders guarantees at least one bidder with marginal valuation distribution G, which is the worst-case highest value. Then, Theorem 3 implies that the second price auction is asymptotically optimal under ambiguity set $\mathcal{F}$.

References

Bei, X., Gravin, N., Lu, P., and Tang, Z. G. (2019). Correlation-robust analysis of single item auction. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 193–208.

Bergemann, D., Brooks, B., and Morris, S. (2016). Informationally robust optimal auction design. mimeo.

Bergemann, D., Brooks, B., and Morris, S. (2017). First-price auctions with general information structures: Implications for bidding and revenue. *Econometrica*, 85(1):107–143.

Bergemann, D., Brooks, B., and Morris, S. (2019). Revenue guarantee equivalence. *American Economic Review*, 109(5):1911–1929.

Bergemann, D. and Morris, S. (2005). Robust mechanism design. *Econometrica*, 73:1771–1813.

Bergemann, D. and Morris, S. (2012). An introduction to robust mechanism design. *Foundation and Trends in Microeconomics*, 8(3):169–230.

Brooks, B. and Du, S. (2021). Optimal auction design with common values: An informationally-robust approach. *Econometrica*, 89(3):1313–1360.

Bulow, J. and Klemperer, P. (1996). Auctions versus negotiations. *American Economic Review*, 86(1):180–194.

Carroll, G. (2017). Robustness and separation in multidimensional screening. *Econometrica*, 85(2):453–488.

Carroll, G. (2019). Robustness in mechanism design and contracting. *Annual Review of Economics*, 11:139–166.

Che, E. W. (2020). Distributionally robust optimal auction design under mean constraints. mimeo.

Che, Y.-K. and Zhong, W. (2021). Robustly-optimal mechanisms for selling multiple goods. mimeo.

Chen, Y.-C. and Li, J. (2018). Revisiting the foundations of dominant-strategy mechanisms. *Journal of Economic Theory*, 178:294–317.

Chung, K.-S. and Ely, J. C. (2007). Foundations of dominant-strategy mechanisms. *Review of Economic Studies*, 74(2):447–476.

Clarke, E. (1971). Multipart pricing of public goods. *Public Choice*, 11(1):17–33.

Cripps, M. W. and Swinkels, J. M. (2006). Efficiency of large double auctions. *Econometrica*, 74(1):47–92.

Du, S. (2018). Robust mechanisms under common valuation. *Econometrica*, 86(5):1569–1588.

Groves, T. (1973). Incentives in teams. *Econometrica*, 41(4):617–631.

He, W. and Li, J. (2022). Correlation-robust auction design. *Journal of Economic Theory*, 200:105403.

Koçyiğit, Ç., Iyengar, G., Kuhn, D., and Wiesemann, W. (2020). Distributionally robust mechanism design. *Management Science*, 66(1):159–189.

Myerson, R. (1981). Optimal auction design. *Mathematics of Operations Research*, 6(1):58–71.

Neeman, Z. (2003). The effectiveness of english auctions. *Games and Economic Behavior*, 43(2):214–238.

Suzdaltsev, A. (2020). An optimal distributionally robust auction. mimeo.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance*, 16(1):8–37.

Yamashita, T. and Zhu, S. (2020). On the foundations of ex post incentive compatible mechanisms. *American Economic Journal: Microeconomics*, Forthcoming.

Zhang, W. (2021). Correlation robustly optimal auctions. mimeo.