Housing markets since Shapley and Scarf

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1 Introduction

Shapley and Scarf (1974) appeared in the first issue of the *Journal of Mathematical Economics*, and is one of the journal's most impactful publications. As we approach the remarkable milestone of the journal's 50th anniversary (1974 - 2024), this article serves as a commemorative exploration of Shapley and Scarf (1974) and the extensive body of literature that follows it.

The contribution of Shapley and Scarf (1974) is threefold. First, this paper embarks on an exploration of solution concepts, such as the core and competitive equilibrium, within an economic framework characterized by the presence of indivisible goods. This distinctive context is exemplified by the housing market. Subsequent studies within this domain have further enriched our understanding of diverse solution concepts in various settings. These investigations delve into aspects such as the existence of solutions, the connections between solution concepts, and the desirable axioms that these solutions satisfy.

Second, their housing market model has established itself as a standard framework for addressing discrete resource allocation problems. The follow-up literature has since explored various extensions of their housing market model. Prominent instances include (i) economies with complex endowments (such as co-ownerships), (ii) economies featuring multiple types of indivisible goods (say, houses and cars), (iii) economies with consumption externalities, etc. These subsequent investigations have significantly broadened our understanding of discrete resource allocation problems.

Third, this paper provides a mechanism/ algorithm, attributed to David Gate, for

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finding competitive prices and core allocations. This mechanism is known as the Top Trading Cycles (TTC). Not only has TTC served as a source of inspiration for numerous market design theories, it has also demonstrated its efficacy through successful applications in various practical contexts, such as housing assignments, school choice systems, and kidney exchange programs, among others.

2 Housing market as per Shapley and Scarf (1974)

This section focuses on the housing market introduced in Shapley and Scarf (1974), discussing different solution concepts, their existence and their relationships.

There are n agents and n houses in the market; the set of agents is $N := \{1, \dots, n\}$ and the set of houses is $H := \{h_1, \dots, h_n\}$. Initially, each agent $i \in N$ owns a house h_i, as endowment. An agent can trade her house for another. Assume that (1) each $i \in N$ has a preference $\succsim_i$ over H, (2) everyone prefers to have a house rather than not having one, and (3) no one ever has use for more than one house (unit demand).

The housing market is denoted by $\Gamma(N, H, \succsim)$. The outcome of the market is a matching $\mu : N \to H$ which assigns each agent $i \in N$ a house $\mu(i) \in H$. For example, the endowment profile is a matching ω such that $\omega(i) = h_i$ for all $i \in N$. Let $\mathcal{M}$ be the set of matchings.

We say that a matching $\mu \in \mathcal{M}$ *(strongly) dominates* another matching $\nu \in \mathcal{M}$ if there exists some coalition $S \subseteq N$ such that

(a) $\{\mu(i) : i \in S\} = \{\omega(i) : i \in S\}$, and

(b) $\mu(i) \succ_i \nu(i)$ for all $i \in S$.

The first condition says that the coalition S can enforce μ among the agents in S, and the second condition says that every member of S strictly prefers μ to ν. The *core* of the housing market is the set of undominated matchings. A matching μ is a *competitive allocation* if there exists a vector of *competitive prices* $p = (p_{h_1}, \dots, p_{h_n}) \in \mathbb{R}_{++}$, one for each house, under which individual optimization decisions of buying and selling will lead to a balance of supply and demand (mathematically, for each i, $p_{\mu(i)} \leq p_{h_i}$ and $\mu(i) \succsim_i h_j$ for all j such that $p_{h_j} \leq p_{\mu(i)}$). In such a case, (μ, p) is referred to as a *competitive equilibrium* (CE).

Shapley and Scarf show that the core of the housing market is always nonempty and David Gale's *top trading cycles (TTC)* algorithm can find a core allocation.[1] Moreover, a vector of competitive prices exists to support the core allocation reached by TTC as a competitive allocation. Now we describe how TTC finds a core allocation and what the competitive prices could be.

For any subset of agents $R \subseteq N$, a *top trading cycle for* R is a nonempty subset $S \subseteq R$ whose $s \geq 1$ members can be indexed in a cyclic order:

$$S = \{i_1, i_2, \ldots, i_s = i_0\},$$

in such a way that $h_{i_{k+1}} \succsim_{i_k} h$ for all $h \in \{h_i : i \in R\}$. A top trading cycle may consist of a single agent; there may be multiple top trading cycles for R. Let S^1 be any top trading cycle for N, S^2 be any top trading cycle for $N - S^1$, S^3 be any top trading cycle for $N - (S^1 \cup S^2)$, and so on until N has been exhausted. By doing so, we have partitioned N into a sequence of one or more disjoint sets:

$$N = S^1 \cup S^2 \cup \cdots \cup S^T.$$

Let μ be the matching such that within each of those top trading cycles, $\mu(i_k) = h_{i_{k+1}}$. Let p be any price vector consisting of $p^1 > p^2 > \cdots > p^T > 0$, one for each top trading cycle, such that for all $i \in S^t$, $p_{h_i} = p^t$.

Theorem 1 (Shapley and Scarf, 1974)**.** *The matching μ is a core allocation; the vector p is a competitive price vector.*

Three insightful examples are provided in the paper, all of which encouraged further investigation of the housing market. The first one demonstrates that if we replace strong dominance with *weak* dominance, i.e., replace (b) with

(b') $\mu(i) \succsim_i \nu(i)$ for all $i \in S$ and $\mu(i) \succ_i \nu(i)$ for some $i \in S$,

then the correspondingly defined *strong* core may be empty. The second example says that a core allocation is not necessarily competitive. This leads to follow up papers on the relationship

[1] Shapley and Scarf's original proof adopted Scarf's theorem (Scarf, 1967). To isolate the impact of Shapley and Scarf (1974) relative to Scarf (1967), we emphasize the housing market it models and what the model brings to the literature.

between core and equilibria. The third example looks at complex preferences: when there are multi-dimensional preferences, the core may be empty.

Shapley and Scarf (1974) stimulate a passion for studying solution concepts in the housing market. Figure 1 shows the relationships between three solutions—the core, the strong core and the competitive equilibria—where CE means the set of competitive allocations and TTC stands for the set of allocations reachable by TTC.[2]

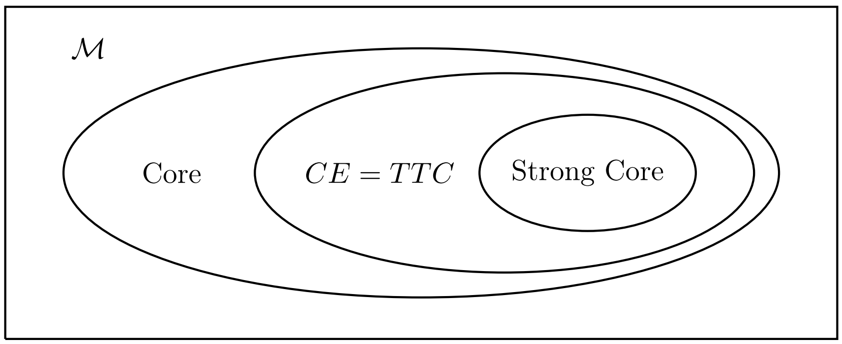

Figure 1: Relationships between solutions.

Some of the relationships are clear from Shapley and Scarf (1974):

1. TTC is contained in the core and it is also contained in CE (both by Theorem 1),
2. the strong core is a strict refinement of the core (strictness by their first example), and
3. a core allocation is not necessarily competitive (by their second example).

However, these facts are not sufficient for drawing the nested structure in Figure 1. Two more facts need be clarified. First, the set of allocations constructed by TTC coincides with the set of competitive allocations, which is thus a strict subset of the core (strictness by fact 3). Second, the strong core is a strict subset of competitive allocations.

For the fact of $CE = TTC$, it suffices to argue that $CE \subseteq TTC$, which is in its essentials

[2] In a slightly generalized setting, Garratt and Qin (1996) introduce lotteries in Shapley and Scarf's housing market. They show similarly that the set of lottery competitive (equilibrium) allocations is nonempty and is contained in the lottery core. In terms of connections with the Shapley and Scarf solutions, they show that the core of the Shapley Scarf model is a strict subset of the lottery core, but the set of lottery competitive (equilibrium) allocations does not contain all competitive allocations. Also see Athanassoglou and Sethuraman (2011) for a generalization of Shapley and Scarf (1974) where arbitrary quantities of each house may be available to the market.

covered in Shapley and Scarf (1974).[3] To see this, take a competitive equilibrium (μ, p) and a house that has the highest price. Tracing the buyer of that house, the buyer of the previous buyer's house, and so on, will result in a trading cycle (possibly of length one). By the definition of competitive equilibrium, the price of the first buyer's house shall be at least as high as the first house. But the first house has the highest price, which implies that the two prices have to be the same. By induction, all houses along the cycle shall have the same price. Since all other houses have lower prices, the trading cycle must be a top trading cycle. Continuing the argument for the rest of houses/agents, we see that (μ, p) could indeed be a TTC-constructed competitive equilibrium. The remaining fact is proved in Wako (1984).[4]

Other than being the finest solution concept, the strong core also satisfies an intuitive selection criterion introduced in Roth and Postlewaite (1977): any allocation μ in the strong core must be in the core of the housing market with μ as the endowment. (This criterion is not satisfied by Shapley and Scarf's original core!) Yet unfortunately, the strong core may be empty (Shapley and Scarf's first example). Roth and Postlewaite (1977) provide the first sufficient condition to guarantee a nonempty strong core.

Theorem 2 (Roth and Postlewaite, 1977)**.** *In the Shapley-Scarf housing market, if no agent is indifferent between any houses, then the strong core is always nonempty, and contains exactly one allocation. This allocation is the unique competitive allocation.*

Thanks to this result, in many theories and applications, three terms could be used interchangeably: *the* strong core allocation, *the* competitive allocation and *the* TTC allocation.[5]

In general settings with indifferences, Quint and Wako (2004) define a condition called segmentability, which means that the set of players can be partitioned into a top trading segmentation. They show that the housing market has a nonempty strong core if and only if

[3] This is claimed in both Roth and Postlewaite (1977) and Wako (1984). Particularly, Roth and Postlewaite (1977, p. 135) mentions that something relevant is in Shapley and Scarf (1974, p. 18). But unfortunately, we couldn't find such a claim in Shapley and Scarf (1974) and the paper does not have a page 18. That is the reason why we are presenting an argument here.

[4] See Wako (1991) for the weak dominance relations between competitive allocations and other allocations.

[5] The unique strong core allocation, when agents' preferences vary, (or the TTC rule) also satisfies further axioms such as *respecting improvement* introduced by Biró et al. (2021) and *swap-flexibility* introduced by Raghavan (2020). Roughly speaking, the former says that if an agent's object becomes more attractive for some other agents, then the agent's allotment in the unique strong core allocation weakly improves, and the latter says that an agent's assigned house under the strong core allocation and another house swap if the two houses' positions swap in the agent's preference.

it is segmentable, and devise an $O(n^3)$ algorithm which returns either a strong core allocation or a report that the strong core is empty.

It is worth emphasizing that the differences between solution concepts in Shapley and Scarf's housing market may not exist in other models with indivisibility. For example, in the assignment game of Shapley and Shubik (1972), an allocation is not just the matching but also involves "side payments." The core in that case is the same as the strong core, and thus also the same as the set of competitive allocations; see also Crawford and Knoer (1981).

Shapley and Scarf's housing market is not the only original economic model with indivisible goods. Some next-of-kin models (but not posterity) are (1) the marriage market and the roommate problem of Gale and Shapley (1962) where men and women are paired, (2) the assignment game of Shapley and Shubik (1972) where indivisible houses are traded between buyers and sellers, and (3) the house allocation model of Hylland and Zeckhauser (1979) where people are assigned to "positions" that are not initially owned by anyone. Since the breakthrough brought by those models, the economics community developed several unified frameworks to accommodate two or more of them, and also uncovered surprising connections between them or with other economic phenomena/concepts/rules. For concrete examples, Quinzii (1984), and more generally Quint (1997), provides unified frameworks for those models above by allowing for monetary endowments and allowing some agents to not have endowed houses; Abdulkadiroğlu and Sönmez (1999) analyzes strategyproof core mechanisms in a house allocation model where there are both existing tenants (Shapley and Scarf, 1974) and new applicants (Hylland and Zeckhauser, 1979);[6] Abdulkadiroğlu and Sönmez (1998), and more generally Carroll (2014), show the in the house allocation problem where agents' preferences are all strict, the *core from random endowment mechanism*—randomly choose an endowment matching with uniform distribution first and then apply TTC in the induced housing market—is equivalent to the *random serial dictatorship* mechanism—randomly determine an ordering and let agents, one by one, choose their top available house.

[6] A *mechanism* maps agents' preference profile into a matching; a *core* mechanism always produce a core allocation; a core mechanism is *strategyproof* if in the induced preference-reporting game, each agent has a dominant strategy of reporting his true preference.

For many other papers focusing more on strategyproofness of mechanisms, such as Ma (1994), Alcalde-Unzu and Molis (2011), Jaramillo and Manjunath (2012), Ehlers (2014), and those on related market design practices, we refer the readers either to later sections or to surveys such as Roth and Sotomayor (1992), Sönmez and Ünver (2011), Abdulkadiroglu and Sönmez (2013), and Roth (2018).

3 Variants of the Shapley-Scarf housing market

The Shapley-Scarf housing market model has established itself as a standard framework for addressing discrete resource allocation problems. The follow-up literature has ventured into diverse expansions of this model. These subsequent inquiries have substantially enriched our comprehension of discrete resource allocation challenges. In this section, we delve into a few of these developments, including: complex endowments (such as co-ownership), multiple types of indivisible goods, and consumption externalities.[7]

Housing markets with complex endowments. There is a striking contrast between the simplicity in endowments in economic models and the complexity of property in practice. A strand of the literature for discrete exchange economics studies exchange economies that place complex endowments—an agent may own multiple goods, none at all, or be a co-owner with others; see, for example, Balbuzanov and Kotowski (2019).

Consider an economy $\Gamma(N, H, \succ, \omega)$, which consists of agents, goods, preferences, and an endowment system. As in Shapley and Scarf (1974), each agent may live in at most one house and each house may take in at most one agent. An allocation $\mu : N \to H \cup \{h_0\}$ is an assignment of agents to houses such that $|\mu^{-1}(h)| \leq 1$ for all $h \in H$, where h_0 is an outside option which has unlimited capacity. Crucially, the endowment system is very general. The endowment system specifies the houses owned by each coalition, and is a function $\omega : 2^N \to 2^H$ satisfying the following properties:

(A1) Agency: $\omega(\emptyset) = \emptyset$.

(A2) Monotonicity: $C' \subseteq C \Rightarrow \omega(C') \subseteq \omega(C)$.

(A3) Exhaustivity: $\omega(N) = H$.

(A4) Non-contestability: For each $h \in H$, there exists $C^h \subseteq N, C^h \neq \emptyset$, such that $h \in \omega(C) \iff C^h \subseteq C$.

[7] The variations we discuss here are mainly within the scope of static models and their solutions. For dynamics in the housing markets, we refer the readers to Serrano and Volij (2008) and Kamijo and Kawasaki (2010), among others. For example, Serrano and Volij (2008) analyse a dynamic trading process where agents may make mistakes with small probability. They provide a foundation for the strong core allocation when no agent is indifferent between any houses. With indifferences, however, the predictions of their dynamic process are not coincide with either the strong core or the set of competitive allocations.

Condition (A1) restricts ownership to agents or groups. Condition (A2) states that a coalition has in its endowment anything that belongs to any sub-coalition. (A3) says that the grand coalition jointly owns everything, and (A4) says that each house has a set of one or more co-owners without opposing and mutually exclusive claims.

For this economy, Balbuzanov and Kotowski (2019) propose a new solution, namely, the exclusion core. The exclusion core rests upon a foundational idea in the legal understanding of property, the right to exclude others. Say that a nonempty coalition $C \subseteq N$ can directly exclusion block the allocation μ with allocation σ if (a) $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and (b) $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \omega(C)$. In words, a coalition can block an assignment whenever each member of the coalition strictly gains and anyone harmed by the reallocation is excluded from a house belonging to the coalition. The direct exclusion core is the set of allocations that cannot be directly exclusion blocked by any nonempty coalition. This shows the immediate power of the right to exclude.

The right to exclude can be more powerful and subtle, as a coalition can inductively relay threats of exclusion and eviction to all agents who are indirectly linked to its endowment $\omega(C)$. The following recursive formulation captures this idea. Let $(\mu^{-1} \circ \omega)(C)$ be the set of agents who are assigned by μ to houses in $\omega(C)$. Thus, with one step of influence, coalition C secures direct and indirect control over $\omega(C_1)$ where $C_1 = C \cup (\mu^{-1} \circ \omega)(C)$. At two steps of influence, it secures control over $\omega(C_2)$ where $C_2 = C_1 \cup (\mu^{-1} \circ \omega)(C_1)$. And so on. The extended endowment of coalition C at allocation μ is $\Omega(C|\omega, \mu) := \omega(\cup_{k=0}^{\infty} C_k)$ where $C_0 = C$ and $C_k = C_{k-1} \cup (\mu^{-1} \circ \omega)(C_{k-1})$ for every $k \geq 1$. A nonempty coalition $C \subseteq N$ can indirectly exclusion block the allocation μ with allocation σ if (a) $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and (b) $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega(C|\omega, \mu)$. The exclusion core is the set of allocation that cannot be indirectly exclusion blocked by any nonempty coalition. Balbuzanov and Kotowski (2019, Theorem 1) establish the existence of the exclusion core for economies that satisfy (A1) - (A4). Exclusion core allocations belong to the direct exclusion core, are Pareto efficient, and also belong to the weak core. Building on Balbuzanov and Kotowski (2019), Sun et al. (2020) and Zhang (2020) propose other variations of the core concept for economies with complex endowments.

Housing markets with multiple types of goods. At the end of the paper Shapley and Scarf (1974), the authors review a series of models involving indivisible goods which have been studied in the literature from the viewpoints of the core, and they conclude: "It would be interesting if a general framework could be found that would unify some of all these scattered results." Quinzii (1984) provides a unified framework by allowing for monetary endowments and allowing some agents to not have endowed houses. More specifically, Quinzii (1984) considers a model of an exchange economy with two goods, where the first good is perfectly divisible (money) and the other good exists only in indivisible units (items). It is assumed that each agent does not initially own more than one indivisible item and has no use for more than one of these items. Quinzii (1984) shows that the associated cooperative game has a nonempty core and that the core allocations coincide with competitive equilibrium allocations under some conditions on the utility functions of the agents. van der Laan et al. (1997) and Yang (2000), among others, generalize this result and establish the existence of competitive equilibria in economies in which there are finitely many different types of indivisible commodities and money.

Konishi et al. (2001) generalize the Shapley-Scarf model by considering multiple types of indivisible goods (but money is not present in the economy). Konishi et al. (2001) show that many of the results from the Shapley-Scarf economy do not carry over to the economy in which two types of indivisible goods are traded, even if the agents' preferences are strict and can be represented by additively separable utility functions. Konishi et al. (2001) also show that the core may be empty in the class of economies with a single type of indivisible good but agents may consume multiple goods (even if no complementarity exists among the goods). Inoue (2008) provides a sufficient condition for the nonemptiness of the weak core in a finite exchange economy where every commodity is available only in integer quantities. Inoue (2008) shows that if the aggregate upper contour set is discretely convex, then the weak core is nonempty.

Housing market with consumption externalities. Externalities in housing markets may take at least three forms.[8] First, the value of a property may depend both on their physical

[8] Sonmez (1999) and Ehlers (2018) consider a class of indivisible goods allocation problems, which includes housing markets as a special case, while also allowing externalities.

attributes (elevation, habitable space, interior design, and geographical exposure) and on the demographics of neighboring homeowners (institutional affiliation, friendships, and other social networks); see, e.g., Baccara et al. (2012) and Massand and Simon (2019).[9] Second, when house trading is only temporary, each agent cares not only about his own assigned house but also about the agent who receives his endowment; see, e.g., Aziz and Lee (2020) and Klaus and Meo (2023). Third, some individuals live in couples and they care not only about their own assigned house but also about the one assigned to their partner; see, e.g., Doğan et al. (2011) and Aslan and Lainé (2020).

To extend the housing market model of Shapley and Scarf (1974) and accommodate externality, one needs to assume that each agent i has a preference R_i *over the matchings* in $\mathcal{M}$ (instead of $\succsim_i$ over individual houses). Let the asymmetric part of R_i be P_i and the set of preference profiles be $\mathcal{R}$. Say that a matching $\mu \in \mathcal{M}$ *(strongly) dominates* another matching $\nu \in \mathcal{M}$ if there exists some coalition $S \subseteq N$ such that

(a) $\{\mu(i) : i \in S\} = \{\omega(i) : i \in S\}$, and

(b) $\mu \; P_i \; \nu$ for all $i \in S$.

The *core* of the housing market with externality is the set of undominated matchings. The refined notion of *strong core* is defined accordingly. The theoretical literature following this line has mostly concentrated on existence of core allocations or other solutions.

Without any restriction on the domain of preferences $\mathcal{R}$, Mumcu and Saglam (2007) shows by a three-agent example that the core may be empty, even if the solution concept is as permissive as defined by strong dominance and agents' preferences are restricted to strict ones (which favors existence in Roth and Postlewaite (1977)). Similar nonexistence results appear in Doğan et al. (2011) and Graziano et al. (2020), among others. These negative result warrants the investigation of either subdomains of $\mathcal{R}$ (specific types of externality in Shapley–Scarf markets), as done by Graziano et al. (2020), Hong and Park (2022), and Klaus and Meo (2023), or the weakening of the solution concept, as done by Baccara et al. (2012) and Massand and Simon (2019).

[9] In a field study of assigning faculty to offices in a new building, which is literally a house allocation and housing market, Baccara et al. (2012) quantify the effects of social network externalities on agents' choices and matching outcomes. Their estimates suggest that network effects have an impact comparable to those of physical attributes.

For an example in the former class, Klaus and Meo (2023) consider housing markets with limited externalities as in the temporary trading case. That is, each agent cares both about his own consumption (traditional "demand preferences") and about the agent who receives his endowment (less traditional "supply preferences"). Suppose both the demand preferences and the supply preferences are strict. Klaus and Meo show that for the demand lexicographic preference domain where agents care primarily about the house they receive (as well as for the supply lexicographic preference domain where agents care primarily about who receives their house), the strong core is nonempty. In the latter class, Baccara et al. (2012) define an assignment (of faculty to offices) as *pairwise stable with transfers* if there is no trade in office assignments between two faculty members that results, with a transfer, in an improvement for both faculty, keeping all other office assignments fixed. Similar pairwise stability, without transfers, is adopted by Massand and Simon (2019) when studying neighbourhood externalities. Both papers provide sufficient conditions to guarantee the existence of their solutions.

4 TTC in Practice

Initiated by Gale and Shapley (1962), matching theory is not only of theoretical interest but also it proves highly beneficial for practical market design issues. This seminal work introduced a *stable marriage* problem where a group of men and women look for a partner from the opposite sex. Each man (woman) has a preference ranking over the women (the men). A matching is *stable* if no unmatched man-woman pair would rather have each other as the partner. They propose a mechanism, so-called "deferred acceptance" (DA) to find a stable matching.

Since then the marriage problem has been adapted to several practical problems, including school choice, housing allocation, and refugee assignment. TTC has also proved to admit desirable properties in these problems, hence, it becomes a main competitor of DA. In the rest of the paper, we pour into these practical market designs and discuss how TTC plays a key role in them.

Before proceeding further, let us describe TTC in a more informal manner to better grasp how it works in markets. Referring to the market sides as objects and agents, each agent points to his best object. Each object, in turn, points to its owner. As both agents and objects are finite, this pointing rule induces cycles. These cycles are then implemented by assigning the agents appearing in them to the objects they are pointing to. The assigned agents and objects are removed from the problem. The same pointing rule is applied in the reduced markets until each agent is assigned an object. The final assignments define the TTC outcome. Gale and Shapley (1962) proposed two DA variants, depending on the side making proposals. Below describes them.

MAN-PROPOSING DEFERRED ACCEPTANCE

Step 1. Each man proposes to his best woman. Each woman tentatively accepts the most preferred man among those proposing herself and rejects the rest.

In general,

Step k. Each rejected man in the previous step proposes to his next best woman. Each woman tentatively accepts the most preferred man among those proposing herself and the tentatively accepted one, and rejects the rest.

The algorithm terminates whenever each man is tentatively accepted by a woman or rejected by all women he would like to match with. The tentative assignments at the terminal round constitute the man-proposing deferred-acceptance outcome. Its another version where women propose to men in the above fashion is called woman-proposing deferred-acceptance.

Gale and Shapley (1962) show that each of these mechanisms always produces a stable matching, hence each is stable. Man (woman)-proposing deferred-acceptance's outcome is unanimously better for the man (woman) side than any other stable matching. Moreover, men (women) cannot manipulate the man (woman)-proposing deferred-acceptance for their sake by preference misreporting. In other words, the former (latter) is strategy-proof for men (women).

Roth (1984) uncovers that DA has already been in use to assign medical interns and residences to hospitals since 1953. This observation has triggered practical matching studies, which have been highly fruitful. It finds applications in several practical problems, including school choice, dorm assignment, housing allocation, kidney exchange, entry-level labor markets, and refugee assignments. In what follows, we mention those for which Shapley and Scarf (1974) play a critical role.

School choice. Balinski and Sönmez (1999) was the first study to adopt matching theory tools for practical college placements. Abdulkadiroğlu and Sönmez (2003) frame elementary and secondary-level student placements as a matching problem, known as school choice. Its basics consist of sets of students and schools. Students have preferences over schools, and schools have priority ranking over students. This is a many-to-one assignment problem as schools have quotas; thus, they can be assigned as many students as at most their quota. Abdulkadiroğlu and Sönmez (2003) discuss three mechanisms: Boston Mechanism (BM), (student-proposing) DA, and TTC. We describe these mechanisms below.

Boston Mechanism (BM)

Step 1. Each student applies to his best school. Each school permanently accepts the top priority students up its quota and rejects the rest.

In general,

Step k. Each rejected student in the previous step applies to his next best school. Each school permanently accepts the top priority applicants up to its remaining capacity and rejects the rest.

The algorithm terminates whenever each student is permanently placed at a school or has already applied to each school. The assignments at the terminal round constitute the BM outcome.

Deferred Acceptance (DA)

Step 1. Each student applies to his best school. Each school tentatively accepts the top priority applicants up to its quota and rejects the rest.

In general,

Step k. Each rejected student in the previous round applies to his next best school. Among the tentatively accepted students and the current step applicants, each school tentatively accepts the top priority students up to its quota and rejects the rest.

The algorithm terminates whenever each student is tentatively accepted by a school or rejected from all schools. The tentative assignments at the terminal round constitute the DA outcome.

TOP TRADING CYCLES (TTC)

Below is a straightforward adaptation of TTC in Shapley and Scarf (1974) to the school choice.

Step 1. Each student points to his top school. Each school points to the top priority student. As everything is finite, there exist cycles. Assign each student in a cycle to the school he is pointing to and decrease the quota of each school by the number of its assigned students.

In general,

Step k. Each student points to his top school with a remaining quota. Each school points to the top priority remaining student. As everything is finite, there exist cycles. Assign each student in a cycle to the school he is pointing to and decrease the quota of each school by the number of its assigned students.

The algorithm terminates whenever each student is assigned a school or schools exhaust all their quotas. The assignments at the terminal round constitute the TTC outcome.

Fairness (also known as "*justified-envy freeness*") is a central concern in school choice. It ensures that no student envies someone else with a lower priority. Fairness, coupled with *individual rationality* and *non-wastefulness*, is referred to as stability.[10] By adapting Gale and Shapley (1962)'s result to school choice, Abdulkadiroğlu and Sönmez (2003) conclude that DA produces the best fair matching for the students.[11] Moreover, DA is immune to preference

[10] Individual rationality ensures that no student would rather be unassigned. Non-wastefulness, on the other hand, eliminates student-school pairs where the student prefers the school and the school has available capacity.

[11] Formally speaking, compared to any other fair matching, DA's outcome is at least weakly preferred by each student.

manipulations by students (Dubins and Freedman, 1981), a property known as "strategy-proofness." However, as identified by Roth (1982), fairness has an efficiency disadvantage in that fairness clashes with (*Pareto*) *efficiency*,[12] implying that DA is not efficient either. Because of this efficiency disadvantage of DA, Abdulkadiroğlu and Sönmez (2003) turn to TTC. They first adapt TTC to the school choice problem, as described above. They then show that TTC is efficient and strategy-proof. However, it is not justified-envy free, in other words, it fails to be fair. The other efficient mechanism they consider is BM, described above. They show that BM is efficient; however, it is not strategy-proof as opposed to TTC. Thus, they mainly promote TTC and DA, leaving the choice between them depending on the priority given to fairness and efficiency properties. If the school choice designer ranks fairness above efficiency, then DA is recommended, and otherwise, TTC is a viable alternative.

The lack of fairness is not a problem specific to TTC because of the existing general trade-off between it and efficiency. However, it still avoids justified envy as much as possible in some senses. When the unfairness level is measured by the set of student-schools pairs, involving in a justified-envy instance, Abdulkadiroğlu et al. (2020) show that TTC turns out to be minimally unfair in the class of efficient and strategy-proof mechanisms in a one-to-one matching setting. This result is then generalized by Doğan and Ehlers (2022), capturing a broader class of measures for the level of unfairness. On the other hand, Kesten (2006) characterize school priorities under which TTC comes to be fair.

The performance of these mechanisms is analyzed in controlled experiments. Chen and Sönmez (2006) experimentally test BM, DA, and TTC in different in school choice. They find that both DA and TTC significantly achieve better efficiency than BM, thus recommending them against BM. Pais and Pinter (2008) test these mechanisms in various informational setups. They find that TTC outperforms both on efficiency ground. It is also better in terms of the truthtelling proportion. Based on these findings, they promote the use of TTC.

Its variants have been proposed in school choice to mitigate the unfairness aspect of TTC. Morrill (2015) introduces two variants of TTC enabling students to receive schools at which they already have top priority without going through a priority trading with others.

[12] A matching is (Pareto) efficient if there is no other matching that is unanimously preferred to the former by students.

Based on the same idea, Hakimov and Kesten (2018) propose another TTC variant. By eliminating particular classes of trading cycles, these TTC variants perform better than TTC on the fairness ground.

TTC has been characterized in the school choice context. Some studies in this direction are Abdulkadiroğlu and Che (2010), Morrill (2013), and Dur and Paiment (2022).

Housing assignment. In housing assignment problems, a bunch of houses are to be distributed among a group of agents. Agents have preferences over houses. Three variants of this problem, depending on the ownership structure, have received attention. Whenever all houses are collectively owned, the problem is called "housing allocation." It is mathematically equivalent to school choice. When each house is owned by some agent, it is called "housing market", studied by Shapley and Scarf (1974). The other variant is a hybrid of them, where some houses are already occupied while others are not (the unoccupied houses are deemed those without an exclusive owner). This section will focus on the last, known as "*House Allocation with Existing Tenants*" introduced by Abdulkadiroğlu and Sönmez (1999).

Agents currently occupying a house are existing tenants. A critical feature of the problem is that the existing tenants often have a right to keep their houses in practice. Hence, they should be ensured to receive at least weakly better houses than what they currently have to participate in the assignment. Because otherwise, they might not risk their current houses by opting out the assignment. This, in turn, would cause a smaller pool of houses, eliminating otherwise beneficial reassignments. Thus, the assignment would not be efficient.

Hence, a necessary condition for achieving efficiency is to guarantee at least weakly better houses than what they currently have to existing tenants. This condition is referred to as "individual rationality." Abdulkadiroğlu and Sönmez (1999) first reveal that some commonly known mechanisms fail to be individually rational or efficient. They then adapt TTC to this problem to ensure individual rationality. It works the same as TTC, with each occupied house pointing to its existing tenant. On the other hand, the unoccupied houses point to the top remaining agent based on a pre-determined ordering. They show that TTC is individually rational, efficient, and strategy-proof. Sönmez and Ünver (2010) then characterize it by these tree properties along with two other requirements.

The theoretical advantages of TTC, identified in Abdulkadiroğlu and Sönmez (1999), are also observed in experiments. Chen and Sönmez (2002) obtain that TTC achieves better efficiency and voluntary participation ratio than its practical competitor, based on serial dictatorship.[13] On the other hand, Guillen and Kesten (2012) conduct a similar analysis, comparing TTC and $NH4$ mechanism, which has been used in MIT for dorm assignments.[14] They report that $NH4$ performs better than TTC regarding both efficiency and truthtelling.

Refugee assignment. Fernández-Huertas Moraga and Rapoport (2014) brought a refugee assignment problem to the matching theory table. Their approach assumes international refugee quota trading. In the absence of quota trading, refugee assignment quite resembles school choice. Delacretaz et al. (2019) is the first to offer a formal theory for the problem, based on the canonical school choice framework.

Following Delacretaz et al. (2019), the problem can be described as follows. There are two disjoint sets: Families and Localities. There is a finite set of dimensions. Each family has a size vector, one for each dimension. Likewise, each locality has a capacity vector, one for each dimension. A matching is an assignment of families to localities such that each family is matched to exactly one locality, and each locality respects its capacity.

Families have preferences over localities, and localities have priorities over families. By adapting TTC to this setup, Delacretaz et al. (2019) introduce two mechanisms, addressing efficiency and strategic desiderata. Andersson and Ehlers (2020) study housing assignments for refugees after their asylums are granted.

Other miscellaneous markets. Dur and Ünver (2019) study matching markets involving two-sided exchanges. Real-life examples include tuition and worker exchanges. The sustainability of such markets depends on the inflow-outflow balances. They adapt TTC to the problem

[13] In a serial dictatorship, each agent chooses his best-remaining alternative one by one following a predetermined ordering. Abdulkadiroğlu and Sönmez (1999) consider a mechanism based on serial dictatorship. It runs the same as serial dictatorship, except the existing tenants have a right to keep their current houses and opt out of the assignment.

[14] In $NH4$, each agent chooses his best remaining house one by one following an order. If an existing tenant prefers his current housing in his turn, it means that someone choosing earlier has already received it. In this case, the existing tenant is assigned his current housing and all the agents' assignments from the latter till the existing tenant are canceled. The process continues starting with these agents whose assignments are canceled, and each agent chooses the best remaining house one by one following the ordering.

and introduce two-sided top-trading-cycles. This mechanism admits desirable properties, including balanced-efficiency and strategy-proofness (for workers).[15]

Combe et al. (2022b) consider teacher assignments in a matching framework. In this problem, teachers and schools compromise the sides of the market. Teachers have preferences over schools, and likewise, schools have preferences over teachers. Teachers have initial assignments a priori. Thus, individual rationality, ensuring no worse assignment than the initial, is a central desideratum. Combe et al. (2022b) introduce a TTC variant mechanism and show that it is two-sided maximal and strategy-proof.[16] Combe et al. (2022a) consider a teach assignment model where some teachers are newcomers, while the rest is tenured. Similar to Combe et al. (2022b), they introduce a TTC-based mechanism and show that it admits desirable strategic, efficiency, and status-quo-improving properties.

TTC has been adopted to markets with distributional constraints, capturing various affirmative action policies. Hafalir et al. (2013) consider minority reserves in school choice and introduce a TTC based mechanism, achieving efficiency and group strategy-proofness. Hamada et al. (2017) introduce a variant of TTC to accommodate minimum quotas and respect initial assignments in school choice. They show that the proposed mechanism is efficient. The authors also conduct a simulation analysis, revealing the advantage of their proposal against a simple extension of TTC. Suzuki et al. (2018) offer another TTC-based algorithm for problems with initial endowments and distributional constraints.

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[15]Balancedness ensures that the export of each college is equal to its import. Balanced-efficiency demands both balancedness and being efficient in the class of balanced matchings.

[16]Two-sided maximality ensures that the matching is individually rational for both sides, and no other matching is unanimously preferred to the former and admitting fewer blocking pairs (in the subset sense).

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